

A Supermartingale Relation for Multivariate Risk Measures

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Abstract

The equivalence between multiportfolio time consistency of a dynamic multivariate risk measure and a supermartingale property is proven. Furthermore, the dual variables under which this set-valued supermartingale is a martingale are characterized as the worst-case dual variables in the dual representation of the risk measure. Examples of multivariate risk measures satisfying the supermartingale property are given.

Keywords: set-valued supermartingale, time consistency, dynamic risk measures, transaction costs, set-valued risk measures, multivariate risks

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1 Introduction

Risk measures, introduced axiomatically in the coherent case in [4, 5] and generalized to the convex case in [27, 29], quantify the minimal capital requirements to cover the risk of a financial portfolio. For their extension to the dynamic, multi-period setting, where the evolution of information known at time t is given by a filtration $(\mathcal{F}_t)_{t=0}^T$, it is natural to ask how the risks relate through time. This led to the definition of time consistency. A risk measure is time consistent if an (almost sure) ordering of risks at a specific time implies the same ordering at all earlier points in time. This property has been studied extensively for scalar valued risks in, e.g., [6, 54, 18, 55, 10, 26, 15, 14, 1, 28] for the discrete time case and [30, 16, 17] for the continuous time case. For the purposes of this paper, we will focus on the equivalence of time consistency and a supermartingale property, which has been studied for coherent risk measures in [6, 12] and for conditionally convex risk measures in [26, 11]. The corresponding result reads

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as: A sequence $(\rho_t)_{t=0}^T$ of sensitive conditionally convex risk measures with minimal penalty function α_t^{min} is time consistent if and only if the process

$$V_t^{\mathbb{Q}}(X) := \rho_t(X) + \alpha_t^{min}(\mathbb{Q})$$

is a \mathbb{Q} -supermartingale, i.e., for any \mathbb{Q} with $\alpha_0^{min}(\mathbb{Q}) < \infty$ and every $X \in L^\infty(\mathbb{R})$,

$$V_t^{\mathbb{Q}}(X) \geq \mathbb{E}^{\mathbb{Q}} \left[V_s^{\mathbb{Q}}(X) \middle| \mathcal{F}_t \right] \quad \mathbb{Q} - a.s.$$

for for all times $0 \leq t < s \leq T$, see [26, Theorem 4.5], [53, Theorem 2.2.2] and [28, Theorem 11.17] for details on the terms and notions. This is an important characterization as it is related to the uniform Doob decomposition under constraints, see [53, Theorem 2.4.6], [25], [28, Chapter 9]; provides a characterization of the time consistent version of a risk measure as the cheapest “hedge” of any $-X \in L^\infty(\mathbb{R})$ in the sense that X is hedged at the terminal time and the incremental costs of that hedging strategy at any time $t + 1$ are acceptable w.r.t. the original risk measure at t , see [53, Proposition 2.5.2]; and provides, e.g., a representation of the superhedging costs under convex trading constraints, see [53, Section 4.2].

In this paper we consider set-valued or multivariate risk measures. Such risk measures and their scalarizations have been of recent interest in the literature. They appear naturally when the random variable whose risk is to be measured is multivariate and not univariate. This is the case, e.g., when multi-asset markets (see e.g. [7, 19]) or markets with frictions (e.g. transaction costs [42, 34, 36, 44, 51, 13] or illiquidity [59]) are considered, or when the components of the random vector represent different risk types or the risks of different units or agents in a group. The latter case gained in particular a lot of attention recently as it allows to study the measurement and regulation of the systemic risk of banking networks, see e.g. [24, 3, 9]. It is also relevant for solvency tests of groups of insurance companies, see [32]. In this multivariate setting one is usually interested in the allocations of the capital charges to the different units, agents, risk types, assets, or currencies. A set-valued risk measure provides these quantities as it assigns to a random vector the set of all capital allocations that compensate its risk. Set-valued risk measures have been studied in a single period framework in, e.g., [42, 37, 34, 36, 13]. Dynamic, multi-period set-valued risk measures have been studied in, e.g., [20, 22, 21, 23, 8]. We will mostly follow the setting and notation of [20, 22] in this paper. In the dynamic multivariate case, a version of time consistency, called multiportfolio time consistency, is used. This property has been shown to be equivalent to a set-valued version of many of the same properties that time consistency is equivalent to in the scalar case.

In this paper we will show that the equivalence of time consistency and a supermartingale property that is well known for scalar dynamic risk measures can be proven in the multivariate case as well. That is, we will show that multiportfolio time consistency of a normalized conditionally convex dynamic risk measures $(R_t)_{t=0}^T$ satisfying certain continuity properties is equivalent to the set-valued stochastic process

$$\mathbb{V}_t^{(\mathbb{Q}, w)}(X) := \text{cl} [R_t(X) + \alpha_t(\mathbb{Q}, w)]$$

being a supermartingale, that is, satisfying for every $(\mathbb{Q}, w) \in \mathcal{W}_t$ and all $X \in L^p(\mathbb{R}^d)$

$$\mathbb{V}_t^{(\mathbb{Q}, w)}(X) \subseteq \text{cl} \mathbb{E}^{\mathbb{Q}} \left[\mathbb{V}_s^{(\mathbb{Q}, w_t^s(\mathbb{Q}, w))}(X) \middle| \mathcal{F}_t \right]$$

for all times $0 \leq t < s \leq T$. Here, α_t denotes the set-valued penalty function of R_t . All terms and notions will be made precise in the main sections of the paper. Note, that as usual in the multivariate setting, one can work on general $L^p(\mathbb{R}^d)$ spaces in contrast to the $L^\infty(\mathbb{R})$ framework of the scalar setting. This is due to the fact that one works with upper sets and not just their boundaries, so the scalar problems stemming from the usage of the essential supremum disappear in the present framework for multivariate risks. The technique to prove the supermartingale results differs drastically from the scalar case. The proofs rely crucially on calculus rules for the (conditional) Minkowski summation and difference as well as representations of convex upper decomposable sets. Furthermore, to complete the proofs we deduce and utilize the dual representations of (conditional) scalarizations of set-valued risk measures – an active field of research, see e.g. [7, 19, 31, 48, 56]. All of these results are proven in the Appendix.

Set-valued sub- and supermartingales are defined e.g. in [52, Chapter 3], [50, Chapter 4], and [41, Chapter 8] for random closed sets and a Doob decomposition is given in [50, Chapter 4.7]. However, due to the ordering relation used here, where smaller risk corresponds to a larger set which thus contains smaller capital requirements, our notion of a supermartingale corresponds to a submartingale in those works.

Characterizations of set-valued supermartingales are highly desirable as set-valued stochastic processes play an important role in many fields of research, e.g. in statistics [58, 49], random set theory [52], and for stochastic differential inclusions [45], with applications to economics and control theory [46, 47]. Furthermore, the obtained result can be seen as a stepping stone towards future research on a set-valued uniform Doob decomposition as well as on hedging of multivariable claims w.r.t. multivariate risk measures.

The paper is structured as follows. In Section 2 we will review properties of dynamic multivariate risk measures from [20, 22] and present a new dual representation for such risk measures. In Section 3 we provide results on the equivalence of multiportfolio time consistency and a set-valued supermartingale property for convex and coherent multivariate risk measures. Finally, in Section 4 we present the main results by extending the results of Section 3, focusing on conditionally convex and conditionally coherent multivariate risk measures. We will provide examples of risk measures satisfying these supermartingale properties. The proofs and intermediate results are collected in the Appendix.

2 Set-valued dynamic risk measures

In this section we will present notation, definitions, and simple results about duality and multiportfolio time consistency for set-valued dynamic risk measures which can be derived from [20, 22].

We will work with a general filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0}^T, \mathbb{P})$ satisfying the usual conditions with $\mathcal{F}_T = \mathcal{F}$. This setting allows for either a discrete time $\{0, 1, \dots, T\}$ or continuous time $[0, T]$ framework. Consider the linear spaces $L_t^p := L^p(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^d)$ for any $p \in [1, \infty]$ and denote $L^p := L_T^p$, where L_t^p is the linear space of the equivalence classes of \mathcal{F}_t -measurable random vectors $X : \Omega \rightarrow \mathbb{R}^d$ with $\|X\|_p^p = \int_\Omega |X(\omega)|^p d\mathbb{P} < \infty$ for $p < \infty$ and $\|X\|_\infty = \text{ess sup}_{\omega \in \Omega} |X(\omega)| < \infty$ for $p = \infty$, where $|\cdot|$ denotes an arbitrary norm in \mathbb{R}^d . We will consider the norm topology on L^p for $p \in [1, \infty)$ and the weak* topology on L^∞ when $p = \infty$. The closure operator cl is

taken as the topological closure throughout this work.

Let $L_t^p(D_t) := \{Z \in L_t^p \mid Z \in D_t \text{ } \mathbb{P}\text{-a.s.}\}$ denote the set of random vectors in L_t^p which take values in D_t \mathbb{P} -a.s. Additionally, throughout this paper we sometimes need to distinguish the spaces of random vectors from those of random variables. To do so, let us denote the linear space of equivalence classes of random variables with finite p -norm $X : \Omega \rightarrow \mathbb{R}$ by $L_t^p(\mathbb{R}) := L^p(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R})$. We will use the following notation for the different types of multiplication: multiplication between a random variable $\lambda \in L^\infty(\mathbb{R})$ and a set of random vectors $D \subseteq L^p$ is defined elementwise $\lambda D = \{\lambda Y \mid Y \in D\} \subseteq L^p$ with $(\lambda Y)(\omega) = \lambda(\omega)Y(\omega)$; the componentwise multiplication between random vectors is denoted by $XY := (X_1Y_1, \dots, X_dY_d)^\top \in L^0$ for $X, Y \in L^0$.

Throughout we will use the notation $L_{t,+}^p := \{X \in L_t^p \mid X \in \mathbb{R}_+^d \text{ } \mathbb{P}\text{-a.s.}\}$ to denote the set of \mathcal{F}_t -measurable random vectors with \mathbb{P} -a.s. non-negative components. Similarly, we will define $L_{t,++}^p := \{X \in L_t^p \mid X \in \mathbb{R}_{++}^d \text{ } \mathbb{P}\text{-a.s.}\}$ as those \mathcal{F}_t -measurable random vectors which are \mathbb{P} -a.s. strictly positive. As with the prior notation, we will define $L_+^p := L_{T,+}^p$ and $L_{++}^p := L_{T,++}^p$. (In)equalities between random vectors (resp. variables) are understood componentwise in the \mathbb{P} -a.s. sense. The set L_+^p defines an ordering on the space of random vectors: $Y \geq X$ for $X, Y \in L^p$ when $Y - X \in L_+^p$.

In financial contexts, a random vector $X \in L_t^p$ represents a portfolio in the sense that component X_i gives the number of units of asset $i \in \{1, \dots, d\}$ held at time t . Thus, we consider portfolios in “physical units” of assets instead of the value of the portfolio in some numéraire. This framework was used and discussed in, e.g., [43, 57, 44].

For risk measurement purposes, fix $m \in \{1, \dots, d\}$ of the assets to be eligible for covering the risk of a portfolio. Without loss of generality we will assume the eligible assets are the first m assets. We will denote by $M := \mathbb{R}^m \times \{0\}^{d-m}$ the subspace of eligible assets. The set of eligible portfolios is given by $M_t := L_t^p(M)$; this is a closed (weak* closed if $p = \infty$) linear subspace of L_t^p (cf. Section 5.4 and Proposition 5.5.1 of [44]). Denote $M_{t,+} := M_t \cap L_{t,+}^p$ to be the non-negative eligible portfolios and $M_{t,-} := -M_{t,+}$ to be the non-positive eligible portfolios.

We are now able to introduce the conditional risk measures as in [20, 22]. A conditional risk measure is a mapping of portfolios (i.e. d -dimensional random vectors) into the upper sets

$$\mathcal{P}(M_t; M_{t,+}) := \{D \subseteq M_t \mid D = D + M_{t,+}\},$$

which is a subset of the power set 2^{M_t} . The output for portfolio X is the set $R_t(X)$ at time t , which is the collection of all eligible portfolios that compensate for the risk.

Definition 2.1. [22, Definition 2.1] A function $R_t : L^p \rightarrow \mathcal{P}(M_t; M_{t,+})$ is a **normalized (conditional) risk measure** at time t if it is

1. M_t -translative: for every $m_t \in M_t : R_t(X + m_t) = R_t(X) - m_t$;
2. L_+^p -monotone: $Y \geq X$ implies $R_t(Y) \supseteq R_t(X)$;
3. finite at zero: $R_t(0) \not\subseteq \{\emptyset, M_t\}$;
4. normalized: for every $X \in L_t^p : R_t(X) = R_t(X) + R_t(0)$.

Additionally, a conditional risk measure at time t is **(conditionally) convex** if for all $X, Y \in L^p$ and all $0 \leq \lambda \leq 1$ ($\lambda \in L_t^\infty(\mathbb{R})$ such that $0 \leq \lambda \leq 1$)

$$R_t(\lambda X + (1 - \lambda)Y) \supseteq \lambda R_t(X) + (1 - \lambda)R_t(Y),$$

it is **(conditionally) positive homogeneous** if for all $X \in L^p$, for all $\lambda > 0$ ($\lambda \in L_t^\infty(\mathbb{R}_{++})$)

$$R_t(\lambda X) = \lambda R_t(X),$$

and is **(conditionally) coherent** if it is (conditionally) convex and (conditionally) positive homogeneous.

A conditional risk measure at time t is **closed** if the graph of the risk measure,

$$\text{graph } R_t = \{(X, u) \in L^p \times M_t \mid u \in R_t(X)\},$$

is closed in the product topology.

A conditional risk measure at time t is **convex upper continuous (c.u.c.)** if for any closed set $D \in \mathcal{G}(M_t; M_{t,-}) := \{D \subseteq M_t \mid D = \text{cl co}(D + M_{t,-})\}$

$$R_t^{-1}(D) := \{X \in L^p \mid R_t(X) \cap D \neq \emptyset\}$$

is closed.

The properties given in Definition 2.1 and their interpretations are discussed in detail in [36, 20, 22]. The image space of a closed convex conditional risk measure is

$$\mathcal{G}(M_t; M_{t,+}) = \{D \subseteq M_t \mid D = \text{cl co}(D + M_{t,+})\}.$$

Note that any c.u.c. risk measure is closed and any closed risk measure has closed images.

A **dynamic risk measure** $(R_t)_{t=0}^T$ is a sequence of conditional risk measures and is said to have one of the properties given in Definition 2.1 if for every time t the conditional risk measure R_t has that property.

For any risk measure R_t there exists an **acceptance set** and vice versa, see Remark 2 and Proposition 2.11 in [20]. For a conditional risk measure R_t the associated acceptance set is defined by the portfolios that require no additional capital to cover the risk, i.e.

$$A_t := \{X \in L^p \mid 0 \in R_t(X)\}.$$

Given an acceptance set A_t , the associated conditional risk measure is defined by the eligible portfolios that, when added to the initial portfolio, make that acceptable, i.e.

$$R_t(X) := \{u \in M_t \mid X + u \in A_t\}.$$

We will define the stepped risk measure and acceptance set by restricting the domain of portfolios to the future eligible assets. That is, for times $0 \leq t < s \leq T$, the stepped risk measure $R_{t,s} : M_s \rightarrow \mathcal{P}(M_t; M_{t,+})$ is defined by $R_{t,s}(X) := R_t(X)$ for any $X \in M_s$ and the stepped acceptance set is defined by $A_{t,s} := \{X \in M_s \mid 0 \in R_t(X)\} = A_t \cap M_s$. We refer to [22, Appendix C] for a detailed discussion of the stepped risk measures.

2.1 Dual representation

In this section we will present the robust representation for conditional risk measures. In [20, 22], a dual representation utilizing the negative convex conjugate, as defined in [33], was given. For this paper, the main results simplify when using the dual

representation w.r.t. the positive convex conjugate introduced in [35]. This dual representation will be deduced below. To provide these results, we will first define the Minkowski subtraction for sets $A, B \subseteq M_t$ by

$$A - \cdot B = \{m \in M_t \mid B + \{m\} \subseteq A\}.$$

The remainder of the setting is identical to that of [20, 22], which we will quickly summarize. Denote the space of d -dimensional probability measures that are absolutely continuous with respect to the physical measure \mathbb{P} by \mathcal{M} . For notational purposes let $\mathcal{M}^e \subseteq \mathcal{M}$ be the set of vector probability measures equivalent to \mathbb{P} . We will consider a \mathbb{P} -a.s. version of the \mathbb{Q} -conditional expectation (for $\mathbb{Q} := (\mathbb{Q}_1, \dots, \mathbb{Q}_d)^\top \in \mathcal{M}$). Let

$$\mathbb{E}^\mathbb{Q}[X \mid \mathcal{F}_t] := \mathbb{E}[\xi_{t,T}(\mathbb{Q})X \mid \mathcal{F}_t]$$

for any $X \in L^p$, where $\xi_{t,s}(\mathbb{Q}) = (\bar{\xi}_{t,s}(\mathbb{Q}_1), \dots, \bar{\xi}_{t,s}(\mathbb{Q}_d))^\top$ for any $0 \leq t \leq s \leq T$ with

$$\bar{\xi}_{t,s}(\mathbb{Q}_i) := \begin{cases} \frac{\mathbb{E}\left[\frac{d\mathbb{Q}_i}{d\mathbb{P}} \mid \mathcal{F}_s\right]}{\mathbb{E}\left[\frac{d\mathbb{Q}_i}{d\mathbb{P}} \mid \mathcal{F}_t\right]} & \text{on } \left\{\omega \in \Omega \mid \mathbb{E}\left[\frac{d\mathbb{Q}_i}{d\mathbb{P}} \mid \mathcal{F}_t\right](\omega) > 0\right\}, \\ 1 & \text{else} \end{cases},$$

see e.g. [14, 20]. Note that for any probability measure $\mathbb{Q}_i \ll \mathbb{P}$ it follows that $\frac{d\mathbb{Q}_i}{d\mathbb{P}} = \bar{\xi}_{0,T}(\mathbb{Q}_i)$. We will say $\mathbb{Q} = \mathbb{P}|_{\mathcal{F}_t}$, i.e. \mathbb{Q} is equal to \mathbb{P} on \mathcal{F}_t , if $\xi_{0,t}(\mathbb{Q}) = (1, \dots, 1)^\top \in \mathbb{R}^d$.

We will denote the half-space $G_t(w)$ and the conditional “half-space” $\Gamma_t(w)$ in L_t^p with normal direction $w \in L_t^q \setminus \{0\}$ by

$$G_t(w) := \left\{u \in L_t^p \mid 0 \leq \mathbb{E}\left[w^\top u\right]\right\}, \quad \Gamma_t(w) := \left\{u \in L_t^p \mid 0 \leq w^\top u\right\}.$$

For ease of notation, for any $w \in L_t^q \setminus \{0\}$ we write

$$G_t^M(w) := G_t(w) \cap M_t, \quad \Gamma_t^M(w) := \Gamma_t(w) \cap M_t.$$

From [22], we can define the set of dual variables at time t to be

$$\mathcal{W}_t = \left\{(\mathbb{Q}, w) \in \mathcal{M} \times \left(M_{t,+}^+ \setminus M_t^\perp\right) \mid w_t^T(\mathbb{Q}, w) \in L_+^q, \mathbb{Q} = \mathbb{P}|_{\mathcal{F}_t}\right\}$$

with

$$w_t^s(\mathbb{Q}, w) := w \xi_{t,s}(\mathbb{Q})$$

for any $0 \leq t \leq s \leq T$. We define the positive dual cone of a cone $C \subseteq L_t^p$ by

$$C^+ = \left\{v \in L_t^q \mid \forall u \in C : \mathbb{E}\left[v^\top u\right] \geq 0\right\}$$

and the orthogonal space of M_t by

$$M_t^\perp = \left\{v \in L_t^q \mid \forall u \in M_t : \mathbb{E}\left[v^\top u\right] = 0\right\}.$$

Corollary 2.2. *A function $R_t : L^p \rightarrow \mathcal{G}(M_t; M_{t,+})$ is a closed convex risk measure if and only if*

$$R_t(X) = \bigcap_{(\mathbb{Q}, w) \in \mathcal{W}_t} \left[\left(\mathbb{E}^\mathbb{Q}[-X \mid \mathcal{F}_t] + G_t(w) \right) \cap M_t - \cdot \beta_t(\mathbb{Q}, w) \right], \quad (2.1)$$

where β_t is the minimal penalty function given by

$$\beta_t(\mathbb{Q}, w) = \bigcap_{Y \in A_t} \left(\mathbb{E}^{\mathbb{Q}}[-Y | \mathcal{F}_t] + G_t(w) \right) \cap M_t. \quad (2.2)$$

R_t is additionally coherent if and only if

$$R_t(X) = \bigcap_{(\mathbb{Q}, w) \in \mathcal{W}_t^{\max}} \left(\mathbb{E}^{\mathbb{Q}}[-X | \mathcal{F}_t] + G_t(w) \right) \cap M_t,$$

for

$$\mathcal{W}_t^{\max} = \{(\mathbb{Q}, w) \in \mathcal{W}_t \mid w_t^T(\mathbb{Q}, w) \in A_t^+\}.$$

Corollary 2.3. *A function $R_t : L^p \rightarrow \mathcal{G}(M_t; M_{t,+})$ is a **closed conditionally convex risk measure** if and only if*

$$R_t(X) = \bigcap_{(\mathbb{Q}, w) \in \mathcal{W}_t} \left[\left(\mathbb{E}^{\mathbb{Q}}[-X | \mathcal{F}_t] + \Gamma_t(w) \right) \cap M_t - \cdot \alpha_t(\mathbb{Q}, w) \right], \quad (2.3)$$

where α_t is the minimal conditional penalty function given by

$$\alpha_t(\mathbb{Q}, w) = \bigcap_{Y \in A_t} \left(\mathbb{E}^{\mathbb{Q}}[-Y | \mathcal{F}_t] + \Gamma_t(w) \right) \cap M_t. \quad (2.4)$$

R_t is additionally conditionally coherent if and only if

$$R_t(X) = \bigcap_{(\mathbb{Q}, w) \in \mathcal{W}_t^{\max}} \left(\mathbb{E}^{\mathbb{Q}}[-X | \mathcal{F}_t] + \Gamma_t(w) \right) \cap M_t. \quad (2.5)$$

For ease of readability, we denote in this paper the positive conjugate by β , respectively α for the conditionally convex case, and the negative conjugate (used in the proofs) by $-\bar{\beta}$ (resp. $-\bar{\alpha}$). This is in contrast to the notation used in [20, 22], where $-\beta$, respectively $-\alpha$, denoted the negative conjugate. The positive and the negative conjugate functions are related to each other by $\beta_t(\mathbb{Q}, w) := G_t^M(w) - \cdot (-\bar{\beta}_t(\mathbb{Q}, w))$, respectively, $\alpha_t(\mathbb{Q}, w) := \Gamma_t^M(w) - \cdot (-\bar{\alpha}_t(\mathbb{Q}, w))$, for any $(\mathbb{Q}, w) \in \mathcal{W}_t$.

2.2 Multiportfolio time consistency

Multiportfolio time consistency has been studied in [20, 22] as a useful concept of time consistency for set-valued risk measures. We will quickly review the definition and some of the equivalent characterizations of this property. In particular, we will provide the cocycle condition on (positive) penalty functions as being equivalent to multiportfolio time consistency as this result will be used in the main proofs of the paper. In contrast, in [22] this result was shown for the negative penalty functions.

Definition 2.4. [22, Definition 2.7] *A dynamic risk measure $(R_t)_{t=0}^T$ is **multiportfolio time consistent** if for all times $0 \leq t < s \leq T$, all portfolios $X \in L^p$ and all sets $\mathbf{Y} \subseteq L^p$ the following implication is satisfied*

$$R_s(X) \subseteq \bigcup_{Y \in \mathbf{Y}} R_s(Y) \Rightarrow R_t(X) \subseteq \bigcup_{Y \in \mathbf{Y}} R_t(Y).$$

Conceptually, a risk measure is multiportfolio time consistent if, whenever any eligible portfolio that compensates for the risk of X will compensate for the risk of some portfolio $Y \in \mathbf{Y}$ at some time, then at any prior time the same relation holds.

Theorem 2.5. [20, Theorem 3.4] *For a normalized dynamic risk measure $(R_t)_{t=0}^T$ the following are equivalent:*

1. $(R_t)_{t=0}^T$ is multiportfolio time consistent,
2. R_t is recursive; that is for all times $0 \leq t < s \leq T$

$$R_t(X) = \bigcup_{Z \in R_s(X)} R_t(-Z) =: R_t(-R_s(X)). \quad (2.6)$$

3. $A_t = A_{t,s} + A_s$ for every time $0 \leq t < s \leq T$.

As shown in the above theorem, multiportfolio time consistency is equivalent to a recursive relation for set-valued risk measures. Furthermore, [23] discusses the relation between the recursive form and a set-valued version of Bellman's principle.

In the case of discrete time $\{0, 1, \dots, T\}$, a step size of 1 (i.e. setting $s = t + 1$) is sufficient to define multiportfolio time consistency and the recursive relation (2.6).

We will now briefly present the cocycle condition for the positive convex conjugates $(\beta_t)_{t=0}^T$ and $(\alpha_t)_{t=0}^T$, which have been proven for the negative conjugates in [22]. Recall from [22] that a conditional risk measure R_t at time t is called conditionally convex upper continuous (c.c.u.c.) if $R_t^{-1}(D) := \{X \in L^p \mid R_t(X) \cap D \neq \emptyset\}$ is closed for any conditionally convex closed set $D \in \mathcal{G}(M_t; M_{t,-})$.

Theorem 2.6. *Let $(R_t)_{t=0}^T$ be a normalized c.u.c. convex risk measure. Then $(R_t)_{t=0}^T$ is multiportfolio time consistent if and only if*

$$\beta_t(\mathbb{Q}, w) = \text{cl} \left(\beta_{t,s}(\mathbb{Q}, w) + \mathbb{E}^{\mathbb{Q}} [\beta_s(\mathbb{Q}, w_t^s(\mathbb{Q}, w)) \mid \mathcal{F}_t] \right)$$

for every $(\mathbb{Q}, w) \in \mathcal{W}_t$ and all times $0 \leq t < s \leq T$.

Theorem 2.7. *Let $(R_t)_{t=0}^T$ be a normalized c.c.u.c. conditionally convex risk measure with dual representation*

$$R_t(X) = \bigcap_{(\mathbb{Q}, w) \in \mathcal{W}_t^c} \left[\left(\mathbb{E}^{\mathbb{Q}} [-X \mid \mathcal{F}_t] + \Gamma_t(w) \right) \cap M_t - \alpha_t(\mathbb{Q}, w) \right]$$

for every $X \in L^p$ where $\mathcal{W}_t^c := \{(\mathbb{Q}, w) \in \mathcal{W}_t \mid \mathbb{Q} \in \mathcal{M}^c\}$. Then $(R_t)_{t=0}^T$ is multiportfolio time consistent if and only if for every $(\mathbb{Q}, w) \in \mathcal{W}_t$ and all times $0 \leq t < s \leq T$

$$\alpha_t(\mathbb{Q}, w) = \text{cl} \left(\alpha_{t,s}(\mathbb{Q}, w) + \mathbb{E}^{\mathbb{Q}} [\alpha_s(\mathbb{Q}, w_t^s(\mathbb{Q}, w)) \mid \mathcal{F}_t] \right).$$

3 Supermartingale Property

In this section we consider a supermartingale-like property for c.u.c. convex set-valued risk measures. This property is akin to that given in [26, 11] for the scalar case.

Let us introduce the following notation

$$V_t^{(\mathbb{Q}, w)}(X) := \text{cl} [R_t(X) + \beta_t(\mathbb{Q}, w)].$$

Theorem 3.1. *Let $(R_t)_{t=0}^T$ be a normalized c.u.c. convex risk measure. $(R_t)_{t=0}^T$ is multiportfolio time consistent if and only if for all times $0 \leq t < s \leq T$ the following supermartingale relation is satisfied: for every $X \in L^p$ and $(\mathbb{Q}, w) \in \mathcal{W}_t$*

$$V_t^{(\mathbb{Q}, w)}(X) \subseteq \mathbb{E}^{\mathbb{Q}} \left[V_s^{(\mathbb{Q}, w_t^s(\mathbb{Q}, w))}(X) \middle| \mathcal{F}_t \right]. \quad (3.1)$$

Furthermore, the assumption on c.u.c. can be weakened on one side of the equivalence: If $(R_t)_{t=0}^T$ is a normalized closed, convex, multiportfolio time consistent risk measure, then (3.1) is satisfied.

Recall from [22] that $\{(\mathbb{Q}, w_t^s(\mathbb{Q}, w)) \mid (\mathbb{Q}, w) \in \mathcal{W}_t\} \subseteq \mathcal{W}_s$ and completely characterizes the dual set \mathcal{W}_s for $t < s$, i.e., for any $(\mathbb{R}, v) \in \mathcal{W}_s$ there exists a $(\mathbb{Q}, w) \in \mathcal{W}_t$ so that for every $X \in L^p$ it follows that $(\mathbb{E}^{\mathbb{Q}}[X | \mathcal{F}_s] + G_s(w_t^s(\mathbb{Q}, w))) \cap M_s = (\mathbb{E}^{\mathbb{R}}[X | \mathcal{F}_s] + G_s(v)) \cap M_s$. As a consequence, the multiportfolio time consistency is equivalent to the supermartingale property of $V_t^{(\mathbb{Q}, w_t^s(\mathbb{Q}, w))}(X)$ holding for all $(\mathbb{Q}, w) \in \mathcal{W}_0$.

Theorem 3.1 will be proven with help of the following two lemmas. The proofs of the lemmas can be found in the Appendix.

Lemma 3.2. *Under the assumptions of Theorem 3.1, the supermartingale relation of Theorem 3.1 holds if and only if the following is satisfied*

$$R_t(X) \supseteq \bigcup_{Z \in R_s(X)} R_t(-Z) \quad (3.2)$$

$$R_t(X) \subseteq \bigcap_{(\mathbb{Q}, w) \in \mathcal{W}_t} \text{cl} \bigcup_{Z \in R_s(X)} \left[\left(\mathbb{E}^{\mathbb{Q}}[Z | \mathcal{F}_t] + G_t(w) \right) \cap M_t - \beta_{t,s}(\mathbb{Q}, w) \right]. \quad (3.3)$$

Lemma 3.3. *Under the assumptions of Theorem 3.1, (3.2), (3.3) are equivalent to*

$$A_t \supseteq A_s + A_{t,s} \quad (3.4)$$

$$A_t \subseteq \bigcap_{(\mathbb{Q}, w) \in \mathcal{W}_t} [A_s + \text{cl} (A_{t,s} + G_s^M(w_t^s(\mathbb{Q}, w)))] . \quad (3.5)$$

Proof of Theorem 3.1. Using Lemmas 3.2 and 3.3 it remains to show that (3.4) and (3.5) are equivalent to multiportfolio time consistency. Clearly, multiportfolio time consistency implies (3.4) and (3.5), see e.g. Theorem 2.5. To prove the converse, let $(R_t)_{t=0}^T$ satisfy (3.4) and (3.5). The crucial observation is that

$$\{w_t^s(\mathbb{Q}, w) \mid (\mathbb{Q}, w) \in \mathcal{W}_t\} = \left\{ \mathbb{E}[Y | \mathcal{F}_s] \mid Y \in L_+^q, \mathbb{E}[Y | \mathcal{F}_t] \notin M_t^\perp \right\},$$

which follows from [20, Lemma 4.5]. Since $M = \mathbb{R}^m \times \{0\}^{d-m}$, $Y \in L_+^q$ implies

$\mathbb{E}[Y|\mathcal{F}_t] \in M_t^\perp$ if and only if $Y \in M_T^\perp$ for any time t . Thus, one obtains

$$\begin{aligned}
A_t &\subseteq \bigcap_{(\mathbb{Q}, w) \in \mathcal{W}_t} [A_s + \text{cl}(A_{t,s} + G_s^M(w_t^s(\mathbb{Q}, w)))] \\
&= \bigcap_{\substack{Y \in L_+^q: \\ \mathbb{E}[Y|\mathcal{F}_t] \notin M_t^\perp}} [A_s + \text{cl}(A_{t,s} + G_s^M(\mathbb{E}[Y|\mathcal{F}_s]))] \\
&= \bigcap_{Y \in L_+^q} [A_s + \text{cl}(A_{t,s} + G_s^M(\mathbb{E}[Y|\mathcal{F}_s]))] \subseteq \bigcap_{Y \in L_+^q} [A_s + \text{cl}(A_{t,s} + G_T(Y))] \\
&\subseteq \bigcap_{Y \in L_+^q} \text{cl}[A_s + A_{t,s} + G_T(Y)] \subseteq \bigcap_{Y \in L_+^q} \text{cl}[\text{cl}(A_s + A_{t,s}) + G_T(Y)] \\
&= \text{cl}(A_s + A_{t,s}) \subseteq \text{cl}(A_t) = A_t.
\end{aligned}$$

Here, the third line follows from $G_s^M(\mathbb{E}[Y|\mathcal{F}_s]) = M_s$ if $\mathbb{E}[Y|\mathcal{F}_s] \in M_s^\perp$ and since

$$A_s + M_s \supseteq \bigcap_{\substack{Y \in L_+^q: \\ \mathbb{E}[Y|\mathcal{F}_s] \notin M_s^\perp}} [A_s + \text{cl}(A_{t,s} + G_s^M(\mathbb{E}[Y|\mathcal{F}_s]))].$$

The last line follows from a separation argument between $\text{cl}(A_s + A_{t,s})$ and $\bigcap_{Y \in L_+^p} \text{cl}[\text{cl}(A_s + A_{t,s}) + G_T(Y)]$ since for any $Y \in L_+^q$

$$\text{cl}[\text{cl}(A_s + A_{t,s}) + G_T(Y)] = \left\{ X \in L^p \mid \mathbb{E}[Y^\top X] \geq \inf_{Z \in \text{cl}(A_s + A_{t,s})} \mathbb{E}[Y^\top Z] \right\}.$$

The final inclusion is directly from (3.4), and the final equality is from A_t closed by assumption of convex upper continuity.

Therefore, $A_t = \text{cl}(A_s + A_{t,s})$, and by [22, Lemma B.4] it follows that $A_s + A_{t,s}$ is closed. Thus $(R_t)_{t=0}^T$ is multiportfolio time consistent by [20, Theorem 3.4].

The last assertion of the theorem holds by noting that the chain of implications from multiportfolio time consistency to (3.4), (3.5) to (3.2), (3.3) to the supermartingale property does not use c.u.c. in addition to closedness. \square

Corollary 3.4. *Let $(R_t)_{t=0}^T$ be a normalized c.u.c. coherent risk measure. $(R_t)_{t=0}^T$ is multiportfolio time consistent if and only if for all times $0 \leq t < s \leq T$*

$$V_t^{(\mathbb{Q}, w)}(X) = \text{cl}[R_t(X) + G_t^M(w)] \subseteq \mathbb{E}^\mathbb{Q} \left[V_s^{(\mathbb{Q}, w_t^s(\mathbb{Q}, w))}(X) \mid \mathcal{F}_t \right]$$

for every $(\mathbb{Q}, w) \in \mathcal{W}_t^{\max}$ and $X \in L^p$. Furthermore, if $(R_t)_{t=0}^T$ is a normalized closed, coherent, multiportfolio time consistent risk measure, then the supermartingale property is satisfied.

Proof. This follows from Theorem 3.1 by noting that, as a consequence of coherence,

$$\beta_t(\mathbb{Q}, w) = \begin{cases} G_t^M(w) & \text{if } (\mathbb{Q}, w) \in \mathcal{W}_t^{\max} \\ \emptyset & \text{else} \end{cases}.$$

If $(\mathbb{Q}, w) \notin \mathcal{W}_t^{\max}$, then $V_t^{(\mathbb{Q}, w)}(X) = \emptyset$ (recalling that $R_t(X) + \emptyset = \emptyset$) and the supermartingale property is trivially satisfied. If $(\mathbb{Q}, w) \in \mathcal{W}_t^{\max}$, then $V_t^{(\mathbb{Q}, w)}(X) = \text{cl}[R_t(X) + G_t^M(w)]$. \square

We will now identify those dual variables that make V_t a martingale as the “worst-case” dual variables in the dual representation. Compare to Proposition 1.21 in [1] for the scalar case.

Corollary 3.5. *Let $(R_t)_{t=0}^T$ be a normalized c.u.c., convex, multiportfolio time consistent risk measure and fix $X \in L^p$. $\left(V_t^{(\mathbb{Q}, w_0^t(\mathbb{Q}, w))}(X)\right)_{t=0}^T$ is a \mathbb{Q} -martingale, i.e.*

$$V_t^{(\mathbb{Q}, w_0^t(\mathbb{Q}, w))}(X) = \mathbb{E}^{\mathbb{Q}} \left[V_s^{(\mathbb{Q}, w_0^s(\mathbb{Q}, w))}(X) \middle| \mathcal{F}_t \right] \quad \forall 0 \leq t < s \leq T,$$

for any $(\mathbb{Q}, w) \in \mathcal{W}_0$ that satisfy the two conditions $\beta_0(\mathbb{Q}, w) \neq \emptyset$ and

$$\text{cl} [R_0(X) + G_0^M(w)] = \left(\mathbb{E}^{\mathbb{Q}} [-X] + G_0(w) \right) \cap M_0 - \beta_0(\mathbb{Q}, w).$$

Additionally, this choice of (\mathbb{Q}, w) is a “worst-case” pair of dual variables for X at any time t , i.e.,

$$\text{cl} [R_t(X) + G_t^M(w_0^t(\mathbb{Q}, w))] = \left(\mathbb{E}^{\mathbb{Q}} [-X | \mathcal{F}_t] + G_t(w_0^t(\mathbb{Q}, w)) \right) \cap M_t - \beta_t(\mathbb{Q}, w_0^t(\mathbb{Q}, w)).$$

If $M = \mathbb{R}^d$ then, conversely, if $\left(V_t^{(\mathbb{Q}, w_0^t(\mathbb{Q}, w))}(X)\right)_{t=0}^T$ is a \mathbb{Q} -martingale for some $(\mathbb{Q}, w) \in \mathcal{W}_0$ with $\beta_0(\mathbb{Q}, w) \neq \emptyset$, then (\mathbb{Q}, w) is a “worst-case” pair of dual variables for X for any time t .

Proof. See appendix, Section B.4. □

Remark 3.6. The supermartingale relation can be given with the negative conjugates $(-\bar{\beta}_t)_{t=0}^T$ (see [20, 22] or Appendix A) though it requires additional considerations due to the fact that $\text{cl} [R_t(X) + G_t^M(w)] - (-\bar{\beta}_t(\mathbb{Q}, w)) \neq \text{cl} [R_t(X) + \beta_t(\mathbb{Q}, w)]$ when $-\bar{\beta}_t(\mathbb{Q}, w) = M_t$ (or equivalently when $\beta_t(\mathbb{Q}, w) = \emptyset$) and $\text{cl} [R_t(X) + G_t^M(w)] = M_t$.

Example 3.7. *Restrictive entropic risk measure:* Consider the full space of eligible assets for all times t , i.e., $M_t = L_t^p$. The restrictive entropic risk measure with parameter $\lambda \in \mathbb{R}_{++}^d$

$$R_t^{\text{ent}}(X) = \left\{ u \in L_t^p \mid \mathbb{E} [1 - \exp(-\lambda(X + u))] \in L_{t,+}^p \right\}$$

is a normalized c.u.c. convex risk measure that is multiportfolio time consistent. For details see [22, Example 3.4 and Section 6.2] and [2]. By Theorem 3.1 we obtain that, with conditional relative entropy $\hat{H}_t(\mathbb{Q}|\mathbb{P}) = \mathbb{E}^{\mathbb{Q}} \left[\log \left(\frac{d\mathbb{Q}}{d\mathbb{P}} \right) \middle| \mathcal{F}_t \right]$,

$$V_t^{(\mathbb{Q}, w)}(X) := \text{cl} \left[R_t^{\text{ent}}(X) + \left(\frac{1}{\lambda} \hat{H}_t(\mathbb{Q}|\mathbb{P}) + G_t(w) \right) \right]$$

is a set-valued supermartingale for any $(\mathbb{Q}, w) \in \mathcal{W}_t$.

Example 3.8. *Composed average value at risk:* Consider a discrete time setting with the full space of eligible assets for all times t , i.e., $M_t = L_t^p$. The average value at risk $AV@R_t(X)$ (for any parameter $\lambda^t \in L_{t,++}^\infty$ bounded away from 0) defines a normalized c.u.c. coherent dynamic risk measure which is not multiportfolio time consistent. However, the composition of the average value at risk $\widetilde{AV@R}_t(X) :=$

$AV@R_t \left(-\widetilde{AV@R_{t+1}}(X) \right)$ is multiportfolio time consistent. For details see [20, Section 5.2], [22, Example 5.5 and Section 6.1], and [38]. By Corollary 3.4,

$$V_t^{(\mathbb{Q}, w)}(X) := \text{cl} \left[\widetilde{AV@R_t}(-X) + G_t(w) \right]$$

is a set-valued supermartingale for any $(\mathbb{Q}, w) \in \widetilde{\mathcal{W}}_t$, where

$$\begin{aligned} \widetilde{\mathcal{W}}_t := \Big\{ (\mathbb{Q}, w) \in \mathcal{W}_t \mid \forall i \in \{1, \dots, d\} \forall s \in \{t, \dots, T-1\} : \\ \mathbb{P} \left(w_i = 0 \text{ or } \bar{\xi}_{s, s+1}(\mathbb{Q}_i) \leq \frac{1}{\lambda_i^t} \right) = 1 \Big\}. \end{aligned}$$

4 Conditional Supermartingale Property

Now we extend the results of the previous section to the conditional penalty function α . That is, we present a supermartingale property for the set-valued stochastic process

$$\mathbb{V}_t^{(\mathbb{Q}, w)}(X) := \text{cl} [R_t(X) + \alpha_t(\mathbb{Q}, w)]$$

for c.u.c. conditionally convex dynamic risk measures $(R_t)_{t=0}^T$.

Corollary 4.1. *Let $(R_t)_{t=0}^T$ be a normalized c.u.c. conditionally convex risk measure with dual representation with equivalent probability measures only, i.e.,*

$$R_t(X) = \bigcap_{(\mathbb{Q}, w) \in \mathcal{W}_t^e} \left[\left(\mathbb{E}^{\mathbb{Q}}[-X \mid \mathcal{F}_t] + \Gamma_t(w) \right) \cap M_t - \alpha_t(\mathbb{Q}, w) \right].$$

Then, $(R_t)_{t=0}^T$ is multiportfolio time consistent if and only if for all times $0 \leq t < s \leq T$

$$\mathbb{V}_t^{(\mathbb{Q}, w)}(X) \subseteq \text{cl} \mathbb{E}^{\mathbb{Q}} \left[\mathbb{V}_s^{(\mathbb{Q}, w_s^s(\mathbb{Q}, w))}(X) \mid \mathcal{F}_t \right] \quad (4.1)$$

for every $(\mathbb{Q}, w) \in \mathcal{W}_t^e$ and $X \in L^p$. Furthermore, if $(R_t)_{t=0}^T$ is a normalized closed, conditionally convex, multiportfolio time consistent risk measure, then the supermartingale property (4.1) is satisfied.

Corollary 4.2. *Let $(R_t)_{t=0}^T$ be a normalized c.u.c. conditionally coherent risk measure with dual representation with equivalent probability measures only, i.e.,*

$$R_t(X) = \bigcap_{(\mathbb{Q}, w) \in \mathcal{W}_t^{e, \max}} \left(\mathbb{E}^{\mathbb{Q}}[-X \mid \mathcal{F}_t] + \Gamma_t(w) \right) \cap M_t$$

where $\mathcal{W}_t^{e, \max} := \mathcal{W}_t^{\max} \cap \mathcal{W}_t^e$. $(R_t)_{t=0}^T$ is multiportfolio time consistent if and only if for all times $0 \leq t < s \leq T$

$$\mathbb{V}_t^{(\mathbb{Q}, w)}(X) = \text{cl} [R_t(X) + \Gamma_t^M(w)] \subseteq \mathbb{E}^{\mathbb{Q}} \left[\mathbb{V}_s^{(\mathbb{Q}, w_s^s(\mathbb{Q}, w))}(X) \mid \mathcal{F}_t \right]$$

for every $(\mathbb{Q}, w) \in \mathcal{W}_t^{e, \max}$. Furthermore, if $(R_t)_{t=0}^T$ is a normalized closed, conditionally coherent, multiportfolio time consistent risk measure, then the supermartingale property is satisfied.

Proof. This follows from Corollary 4.1 since

$$\alpha_t(\mathbb{Q}, w) = \begin{cases} \Gamma_t^M(w) & \text{if } (\mathbb{Q}, w) \in \mathcal{W}_t^{\max} \\ \emptyset & \text{else} \end{cases}$$

and thus $\mathbb{V}_t^{(\mathbb{Q}, w)}(X) = \text{cl} [R_t(X) + \Gamma_t^M(w)]$ for any $(\mathbb{Q}, w) \in \mathcal{W}_t^{\max}$. \square

Again, we can characterize the dual variables under which \mathbb{V}_t is a martingale as the “worst-case” dual variables.

Corollary 4.3. *Let $(R_t)_{t=0}^T$ be a normalized c.u.c., conditionally convex, multiportfolio time consistent risk measure with dual representation with equivalent probability measures only, i.e.,*

$$R_t(X) = \bigcap_{(\mathbb{Q}, w) \in \mathcal{W}_t^e} \left[\left(\mathbb{E}^{\mathbb{Q}}[-X | \mathcal{F}_t] + \Gamma_t(w) \right) \cap M_t - \cdot \alpha_t(\mathbb{Q}, w) \right].$$

Fix some $X \in L^P$. For any $(\mathbb{Q}, w) \in \mathcal{W}_0^e$ satisfying $\alpha_0(\mathbb{Q}, w) \neq \emptyset$ and

$$\text{cl} [R_0(X) + \Gamma_0^M(w)] = \left(\mathbb{E}^{\mathbb{Q}}[-X] + \Gamma_0(w) \right) \cap M_0 - \cdot \alpha_0(\mathbb{Q}, w),$$

the stochastic process $\left(V_t^{(\mathbb{Q}, w_0^t(\mathbb{Q}, w))}(X) \right)_{t=0}^T$ is a \mathbb{Q} -martingale, i.e.

$$\mathbb{V}_t^{(\mathbb{Q}, w_0^t(\mathbb{Q}, w))}(X) = \text{cl} \mathbb{E}^{\mathbb{Q}} \left[\mathbb{V}_s^{(\mathbb{Q}, w_s^s(\mathbb{Q}, w))}(X) \middle| \mathcal{F}_t \right] \quad \forall 0 \leq t < s \leq T.$$

Additionally, this choice of (\mathbb{Q}, w) is a “worst-case” pair of dual variables for X at any time t , i.e.,

$$\text{cl} [R_t(X) + \Gamma_t^M(w_0^t(\mathbb{Q}, w))] = \left(\mathbb{E}^{\mathbb{Q}}[-X | \mathcal{F}_t] + \Gamma_t(w_0^t(\mathbb{Q}, w)) \right) \cap M_t - \cdot \alpha_t(\mathbb{Q}, w_0^t(\mathbb{Q}, w)).$$

If $M = \mathbb{R}^d$ then, conversely, if $\left(V_t^{(\mathbb{Q}, w_0^t(\mathbb{Q}, w))}(X) \right)_{t=0}^T$ is a \mathbb{Q} -martingale for some $(\mathbb{Q}, w) \in \mathcal{W}_0^e$ with $\alpha_0(\mathbb{Q}, w) \neq \emptyset$ then (\mathbb{Q}, w) is a “worst-case” pair of dual variables for X for any time t .

Example 4.4. Superhedging: Consider a discrete time setting with the full space of eligible assets for all times t , i.e., $M_t = L_t^P$. Define a market with convex transaction costs described by convex solvency regions $(K_t)_{t=0}^T$ with

$$\text{int recc}(K_t[\omega]) \supseteq \mathbb{R}_+^d \setminus \{0\}$$

almost surely, where $\text{recc}(C)$ denotes the recession cone of a convex set $C \subseteq \mathbb{R}^d$. The set of superhedging portfolios $SHP_t(X)$ at time t denotes those eligible portfolios that can be traded from time t to the terminal time T to outperform the input portfolio $X \in L^P$. From this formulation we can immediately define a closed, conditionally convex multivariate risk measure $R_t(X) := SHP_t(-X)$ which is multiportfolio time consistent. For details see [22, Example 5.4]. From the constraint on the interior of the solvency regions, the penalty function is only defined on the set of equivalent probability measures. Though SHP_t is not normalized in general, we may still apply

the results of this section as the summation of acceptance sets and penalty function representations holds via a composition backwards in time (see [22, Section 5]) and thus all proofs follow accordingly. By Corollary 4.1 we obtain that

$$\mathbb{V}_t^{(\mathbb{Q}, w)}(X) := \text{cl} [SHP_t(-X) + \alpha_t^{SHP}(\mathbb{Q}, w)]$$

defines a set-valued supermartingale for any $(\mathbb{Q}, w) \in \mathcal{W}_t^e$, where the penalty function is given by

$$\alpha_t^{SHP}(\mathbb{Q}, w) := \sum_{s=t}^T \left\{ u \in L_t^p \mid \text{ess sup}_{k_s \in L_s^p(K_s)} -w^\top \mathbb{E}^\mathbb{Q}[k_s | \mathcal{F}_t] \leq w^\top u \right\}.$$

In the special case that the market has proportional transaction costs only, i.e., the market is defined by a sequence of solvency cones $(K_t)_{t=0}^T$, then the set of superhedging portfolios defines a normalized, closed, conditionally coherent risk measure which is multiperfolio time consistent. For details see [20, Section 5.1], [22, Example 4.7], and [51]. Then, by Corollary 4.2 we obtain that

$$\mathbb{V}_t^{(\mathbb{Q}, w)}(X) := \text{cl} [SHP_t(-X) + \Gamma_t(w)]$$

defines a set-valued supermartingale for any $(\mathbb{Q}, w) \in \mathcal{W}_{\{t, \dots, T\}}$, where

$$\mathcal{W}_{\{t, \dots, T\}} := \{(\mathbb{Q}, w) \in \mathcal{W}_t^e \mid w_t^s(\mathbb{Q}, w) \in L_s^q(K_s^+) \forall s \in \{t, \dots, T\}\}.$$

Similarly we could apply Theorem 3.1 (Corollary 3.4 in the case of proportional transaction costs), which yields that $V_t^{(\mathbb{Q}, w)}(X)$ is a set-valued supermartingale as well.

A Proofs for Section 2

A.1 Proof of Corollary 2.2

Proof. Recall from [22, Theorem 2.3] that a closed convex risk measure R_t has the dual representation

$$R_t(X) = \bigcap_{(\mathbb{Q}, w) \in \mathcal{W}_t} \left[-\bar{\beta}_t(\mathbb{Q}, w) + \left(\mathbb{E}^\mathbb{Q}[-X | \mathcal{F}_t] + G_t(w) \right) \cap M_t \right], \quad (\text{A.1})$$

where $-\bar{\beta}_t$ is the minimal penalty function given by

$$-\bar{\beta}_t(\mathbb{Q}, w) = \text{cl} \bigcup_{Y \in A_t} \left(\mathbb{E}^\mathbb{Q}[Z | \mathcal{F}_t] + G_t(w) \right) \cap M_t. \quad (\text{A.2})$$

The dual variables are in the set \mathcal{W}_t , which is independent of a dual representation with positive or negative conjugates. This immediately yields (2.1) by biconjugation with positive conjugates as in [35, Theorem 5.8]. Therefore, it remains to show that $\beta_t(\mathbb{Q}, w) = \bigcap_{Y \in A_t} (\mathbb{E}^\mathbb{Q}[-Y | \mathcal{F}_t] + G_t(w)) \cap M_t$. Using the relationship $\beta_t(\mathbb{Q}, w) := G_t^M(w) - \cdot (-\bar{\beta}_t(\mathbb{Q}, w))$ for any $(\mathbb{Q}, w) \in \mathcal{W}_t$ between the positive and negative conjugate

function as given in [35, Remark 5.5] and the expression (2.2) for the negative conjugate, one obtains

$$\begin{aligned}
u \in \beta_t(\mathbb{Q}, w) &= G_t^M(w) - \cdot (-\bar{\beta}_t(\mathbb{Q}, w)) \Leftrightarrow -\bar{\beta}_t(\mathbb{Q}, w) + u \subseteq G_t^M(w) \\
&\Leftrightarrow u \in M_t, \mathbb{E} \left[w^\top (b + u) \right] \geq 0 \quad \forall b \in -\bar{\beta}_t(\mathbb{Q}, w) \\
&\Leftrightarrow u \in M_t, \mathbb{E} \left[w^\top u \right] \geq \mathbb{E} \left[w^\top \mathbb{E}^\mathbb{Q}[-Y | \mathcal{F}_t] \right] \quad \forall Y \in A_t \\
&\Leftrightarrow u \in \left(\mathbb{E}^\mathbb{Q}[-Y | \mathcal{F}_t] + G_t(w) \right) \cap M_t \quad \forall Y \in A_t \\
&\Leftrightarrow u \in \bigcap_{Y \in A_t} \left(\mathbb{E}^\mathbb{Q}[-Y | \mathcal{F}_t] + G_t(w) \right) \cap M_t,
\end{aligned}$$

where the third line uses $-\bar{\beta}_t(\mathbb{Q}, w) = \{b \in M_t \mid \mathbb{E}[w^\top b] \geq \inf_{Y \in A_t} \mathbb{E}[w^\top \mathbb{E}^\mathbb{Q}[Y | \mathcal{F}_t]]\}$, M_t being a linear space and the continuity of the linear operator. The coherent case corresponds to [22, Theorem 2.3]. \square

A.2 Proof of Corollary 2.3

We will use the following proposition.

Proposition A.1. $\text{cl} [A + \Gamma_t^M(w)] = \{m \in M_t \mid w^\top m \geq \text{ess inf}_{a \in A} w^\top a\}$ if $A \subseteq M_t$ is convex and decomposable.

Proof. First, it trivially can be seen that

$$D := \left\{ m \in M_t \mid w^\top m \geq \text{ess inf}_{a \in A} w^\top a \right\} = \overline{\text{cl}} [A + \Gamma_t^M(w)]$$

where $\overline{\text{cl}}$ is the closure with respect to the almost sure convergence of the set of random vectors.

Second, we will show that $\text{cl} [A + \Gamma_t^M(w)] \subseteq D$ (recalling that the closure operator cl is w.r.t. the topological closure). We will break this up into two cases: $p \in [1, \infty)$ and $p = \infty$.

Consider $p < \infty$. To prove this statement we will show that D is closed in the norm topology. Let $(m_n)_{n \in \mathbb{N}} \rightarrow m \in M_t$ converge in the p -norm for $m_n \in D$ for every $n \in \mathbb{N}$. Since p -norm convergence implies convergence in probability, we know that there exists a subsequence $(m_{n_k})_{k \in \mathbb{N}} \rightarrow m$ which converges almost surely, thus $m \in D$. This implies that $\text{cl} [A + \Gamma_t^M(w)] \subseteq D$.

Consider $p = \infty$. To prove this statement we will show that D is weak* closed. Let $(m_i)_{i \in I} \rightarrow m \in M_t$ in the weak* topology so that $m_i \in D$ for all $i \in I$. Let $\Delta := \{\omega \in \Omega \mid w(\omega)^\top m(\omega) < [\text{ess inf}_{a \in A} w^\top a](\omega)\} \in \mathcal{F}_t$. Assume $\mathbb{P}(\Delta) > 0$, immediately it follows that

$$\mathbb{E} \left[1_\Delta \text{ess inf}_{a \in A} w^\top a \right] \leq \liminf_{i \in I} \mathbb{E} \left[1_\Delta w^\top m_i \right] = \mathbb{E} \left[1_\Delta w^\top m \right] < \mathbb{E} \left[1_\Delta \text{ess inf}_{a \in A} w^\top a \right].$$

By contradiction this implies $\mathbb{P}(\Delta) = 0$ and thus $w^\top m \geq \text{ess inf}_{a \in A} w^\top a$, i.e. D is weak* closed. This implies that $\text{cl} [A + \Gamma_t^M(w)] \subseteq D$.

Now we will show that $\text{cl}[A + \Gamma_t^M(w)] \supseteq D$. Assume this is not true, i.e. let $u \in D$ such that $u \notin \text{cl}[A + \Gamma_t^M(w)]$. By a separation argument there exists $v \in L_t^q$ such that

$$\mathbb{E}[v^\top u] < \inf_{m \in \text{cl}[A + \Gamma_t^M(w)]} \mathbb{E}[v^\top m].$$

By construction of $\Gamma_t^M(w)$,

$$\inf_{m \in \text{cl}[A + \Gamma_t^M(w)]} \mathbb{E}[v^\top m] = \begin{cases} \inf_{a \in A} \mathbb{E}[\lambda w^\top a] & \text{if } v = \lambda w \text{ for some } \lambda \geq 0 \text{ a.s.} \\ -\infty & \text{else} \end{cases}.$$

Noting that we can exchange the expectation and infimum due to decomposability (cf. [60, Theorem 1]),

$$\mathbb{E}[\lambda w^\top u] < \inf_{a \in A} \mathbb{E}[\lambda w^\top a] = \mathbb{E}\left[\text{ess inf}_{a \in A} \lambda w^\top a\right] = \mathbb{E}\left[\lambda \text{ess inf}_{a \in A} w^\top a\right].$$

However this contradicts $u \in D$. Thus $\text{cl}[A + \Gamma_t^M(w)] \supseteq D$. \square

Proof of Corollary 2.3. First, by using Proposition A.1, we can reformulate the conditional penalty function as

$$\alpha_t(\mathbb{Q}, w) = \left\{ u \in M_t \mid w^\top u \geq \text{ess sup}_{Y \in A_t} w^\top \mathbb{E}^\mathbb{Q}[-Y \mid \mathcal{F}_t] \right\}. \quad (\text{A.3})$$

So, for any $(\mathbb{Q}, w) \in \mathcal{W}_t$

$$\begin{aligned} & \left(\mathbb{E}^\mathbb{Q}[-X \mid \mathcal{F}_t] + \Gamma_t(w) \right) \cap M_t - \alpha_t(\mathbb{Q}, w) \\ &= \left\{ u \in M_t \mid u + \alpha_t(\mathbb{Q}, w) \subseteq \left(\mathbb{E}^\mathbb{Q}[-X \mid \mathcal{F}_t] + \Gamma_t(w) \right) \cap M_t \right\} \\ &= \left\{ u \in M_t \mid \left\{ m \in M_t \mid w^\top m \geq w^\top u + \text{ess sup}_{Y \in A_t} w^\top \mathbb{E}^\mathbb{Q}[-Y \mid \mathcal{F}_t] \right\} \right. \\ & \quad \left. \subseteq \left\{ m \in M_t \mid w^\top m \geq w^\top \mathbb{E}^\mathbb{Q}[-X \mid \mathcal{F}_t] \right\} \right\} \\ &= \left\{ u \in M_t \mid w^\top u + \text{ess sup}_{Y \in A_t} w^\top \mathbb{E}^\mathbb{Q}[-Y \mid \mathcal{F}_t] \geq w^\top \mathbb{E}^\mathbb{Q}[-X \mid \mathcal{F}_t] \right\} \\ &= \left\{ u \in M_t \mid w^\top u \geq \text{ess inf}_{Y \in A_t} w^\top \mathbb{E}^\mathbb{Q}[Y - X \mid \mathcal{F}_t] \right\} \\ &= -\bar{\alpha}_t(\mathbb{Q}, w) + \left(\mathbb{E}^\mathbb{Q}[-X \mid \mathcal{F}_t] + \Gamma_t(w) \right) \cap M_t. \end{aligned}$$

Thus, now the result follows directly from the dual representation [22, Theorem 2.3] w.r.t. the negative conjugate function. The above chain of equalities follow via the definition of the Minkowski subtraction, the result of Proposition A.1, reformulating the inclusion, and by the definition of the negative conjugate function from [22] respectively. The conditionally coherent case is as in [22, Corollary 2.4]. \square

A.3 Proof of Theorem 2.6

The proof of Theorem 2.6 involves the following two propositions.

Proposition A.2. *Consider $w \in M_{t,+}^+ \setminus M_t^\perp$. For any sets $A, B \in \mathcal{G}(M_t; M_{t,+})$ such that $A \neq \emptyset$ if $\text{cl}[B + G_t^M(w)] = M_t$, and $B \neq \emptyset$ if $\text{cl}[A + G_t^M(w)] = M_t$, it holds that*

$$G_t^M(w) - \cdot \text{cl}[A + B] = \text{cl}[(G_t^M(w) - \cdot A) + (G_t^M(w) - \cdot B)]. \quad (\text{A.4})$$

Proof. “ \supseteq ” If $\text{cl}[(G_t^M(w) - \cdot A) + (G_t^M(w) - \cdot B)] = \emptyset$, the inclusion is trivial. So, let us assume the right hand side of (A.4) is nonempty and consider an element $u \in \text{cl}[(G_t^M(w) - \cdot A) + (G_t^M(w) - \cdot B)]$. Then there exists a net $(u_i^A)_{i \in I}$ and $(u_j^B)_{j \in J}$ such that $u = \lim_{i,j} (u_i^A + u_j^B)$ and $u_i^A \in G_t^M(w) - \cdot A$ and $u_j^B \in G_t^M(w) - \cdot B$ for every i, j . Immediately, by definition of $- \cdot$ we obtain

$$u_i^A + u_j^B + \text{cl}[A + B] \subseteq G_t^M(w) + G_t^M(w) = G_t^M(w).$$

Therefore, $u_i^A + u_j^B \in G_t^M(w) - \cdot \text{cl}[A + B]$ for every i, j . Since $G_t^M(w) - \cdot \text{cl}[A + B]$ is closed by definition, one has $u \in G_t^M(w) - \cdot \text{cl}[A + B]$.

“ \subseteq ” If $G_t^M(w) - \cdot \text{cl}[A + B] = \emptyset$, the inclusion is trivial. So, let us assume the left hand side of (A.4) is nonempty and consider $u \in G_t^M(w) - \cdot \text{cl}[A + B]$. By definition this is equivalent to $\mathbb{E}[w^\top u] + \inf_{a \in A} \mathbb{E}[w^\top a] + \inf_{b \in B} \mathbb{E}[w^\top b] \geq 0$ and $u \in M_t$. First consider the case where $\inf_{b \in B} \mathbb{E}[w^\top b] \in \mathbb{R}$. Let $u^B \in M_t$ so that $\mathbb{E}[w^\top u^B] = -\inf_{b \in B} \mathbb{E}[w^\top b]$, where the existence of u^B is guaranteed by the continuity of the linear operator, M_t being a linear space, and $w \notin M_t^\perp$. Define $u^A := u - u^B \in M_t$. By construction $u^A \in G_t^M(w) - \cdot A$, $u^B \in G_t^M(w) - \cdot B$, and $u = u^A + u^B$. That is, $u \in \text{cl}[(G_t^M(w) - \cdot A) + (G_t^M(w) - \cdot B)]$. If $\inf_{b \in B} \mathbb{E}[w^\top b] = \infty$ then by assumption $\inf_{a \in A} \mathbb{E}[w^\top a] > -\infty$, so $G_t^M(w) - \cdot \text{cl}[A + B] = M_t$, $G_t^M(w) - \cdot A \neq \emptyset$, and $G_t^M(w) - \cdot B = M_t$ and thus the inclusion trivially holds. The case $\inf_{b \in B} \mathbb{E}[w^\top b] = -\infty$ cannot occur under the current assumption of the left hand side of (A.4) being nonempty as it would imply $\inf_{a \in A} \mathbb{E}[w^\top a] < \infty$ by assumption and therefore $G_t^M(w) - \cdot \text{cl}[A + B] = \emptyset$. \square

Proposition A.3. *Let $s > t$ and $(\mathbb{Q}, w) \in \mathcal{W}_t$. For any set $A \in \mathcal{G}(M_s; M_{s,+})$ it holds*

$$G_t^M(w) - \cdot \mathbb{E}^\mathbb{Q}[A | \mathcal{F}_t] = \mathbb{E}^\mathbb{Q}[G_s^M(w_t^s(\mathbb{Q}, w)) - \cdot A | \mathcal{F}_t].$$

Proof.

$$\begin{aligned} \mathbb{E}^\mathbb{Q}[G_s^M(w_t^s(\mathbb{Q}, w)) - \cdot A | \mathcal{F}_t] &= \left\{ \mathbb{E}^\mathbb{Q}[u | \mathcal{F}_t] \mid u \in M_s, u + A \subseteq G_s^M(w_t^s(\mathbb{Q}, w)) \right\} \\ &= \left\{ \mathbb{E}^\mathbb{Q}[u | \mathcal{F}_t] \mid u \in M_s, \inf_{a \in A} \mathbb{E}[w_t^s(\mathbb{Q}, w)^\top (u + a)] \geq 0 \right\} \\ &= \left\{ \mathbb{E}^\mathbb{Q}[u | \mathcal{F}_t] \mid u \in M_s, \inf_{a \in A} \mathbb{E}[w^\top \mathbb{E}^\mathbb{Q}[u + a | \mathcal{F}_t]] \geq 0 \right\} \\ &= \left\{ u \in M_t \mid \inf_{a \in A} \mathbb{E}[w^\top (\mathbb{E}^\mathbb{Q}[a | \mathcal{F}_t] + u)] \geq 0 \right\} \\ &= \left\{ u \in M_t \mid u + \mathbb{E}^\mathbb{Q}[A | \mathcal{F}_t] \subseteq G_t^M(w) \right\} = G_t^M(w) - \cdot \mathbb{E}^\mathbb{Q}[A | \mathcal{F}_t]. \end{aligned}$$

\square

Proof of Theorem 2.6. Recall from [35, Remark 5.5] that the conjugate and negative conjugate are related via $\beta_t(\mathbb{Q}, w) = G_t^M(w) - \cdot (-\bar{\beta}_t(\mathbb{Q}, w))$ and $-\bar{\beta}_t(\mathbb{Q}, w) = G_t^M(w) - \cdot \beta_t(\mathbb{Q}, w)$ for every $(\mathbb{Q}, w) \in \mathcal{W}_t$. Recall from [22, Theorem 3.2] that multiperiod time consistency is equivalent to the cocycle condition on the negative conjugates, i.e.,

$$-\bar{\beta}_t(\mathbb{Q}, w) = \text{cl} \left(-\bar{\beta}_{t,s}(\mathbb{Q}, w) + \mathbb{E}^{\mathbb{Q}} [-\bar{\beta}_s(\mathbb{Q}, w_t^s(\mathbb{Q}, w)) | \mathcal{F}_t] \right)$$

for every $(\mathbb{Q}, w) \in \mathcal{W}_t$ and all times $0 \leq t < s \leq T$. Let $(\mathbb{Q}, w) \in \mathcal{W}_t$.

“ \Rightarrow ” Since $(R_t)_{t=0}^T$ is multiperiod time consistent, we obtain

$$\begin{aligned} \beta_t(\mathbb{Q}, w) &= G_t^M(w) - \cdot (-\bar{\beta}_t(\mathbb{Q}, w)) \\ &= G_t^M(w) - \cdot \text{cl} \left(-\bar{\beta}_{t,s}(\mathbb{Q}, w) + \mathbb{E}^{\mathbb{Q}} [-\bar{\beta}_s(\mathbb{Q}, w_t^s(\mathbb{Q}, w)) | \mathcal{F}_t] \right) \\ &= \text{cl} \left[\left(G_t^M(w) - \cdot (-\bar{\beta}_{t,s}(\mathbb{Q}, w)) \right) + \left(G_t^M(w) - \cdot \mathbb{E}^{\mathbb{Q}} [-\bar{\beta}_s(\mathbb{Q}, w_t^s(\mathbb{Q}, w)) | \mathcal{F}_t] \right) \right] \\ &= \text{cl} \left[\beta_{t,s}(\mathbb{Q}, w) + \left(\mathbb{E}^{\mathbb{Q}} [G_s^M(w_t^s(\mathbb{Q}, w) - \cdot (-\bar{\beta}_s(\mathbb{Q}, w_t^s(\mathbb{Q}, w))) | \mathcal{F}_t] \right) \right] \\ &= \text{cl} \left[\beta_{t,s}(\mathbb{Q}, w) + \mathbb{E}^{\mathbb{Q}} [\beta_s(\mathbb{Q}, w_t^s(\mathbb{Q}, w)) | \mathcal{F}_t] \right], \end{aligned}$$

where the third and fourth equations follow from Proposition A.2 and Proposition A.3, respectively. The assumptions of Proposition A.2 are satisfied as $-\bar{\beta}_t(\mathbb{Q}, w) \neq \emptyset$ for $(\mathbb{Q}, w) \in \mathcal{W}_t$ and thus $A := -\bar{\beta}_{t,s}(\mathbb{Q}, w) \neq \emptyset$ and $B := \mathbb{E}^{\mathbb{Q}} [-\bar{\beta}_s(\mathbb{Q}, w_t^s(\mathbb{Q}, w)) | \mathcal{F}_t] \neq \emptyset$.

“ \Leftarrow ” Since $\beta_t(\mathbb{Q}, w) = \text{cl} (\beta_{t,s}(\mathbb{Q}, w) + \mathbb{E}^{\mathbb{Q}} [\beta_s(\mathbb{Q}, w_t^s(\mathbb{Q}, w)) | \mathcal{F}_t])$, it holds

$$\begin{aligned} -\bar{\beta}_t(\mathbb{Q}, w) &= G_t^M(w) - \cdot \beta_t(\mathbb{Q}, w) \\ &= G_t^M(w) - \cdot \text{cl} \left(\beta_{t,s}(\mathbb{Q}, w) + \mathbb{E}^{\mathbb{Q}} [\beta_s(\mathbb{Q}, w_t^s(\mathbb{Q}, w)) | \mathcal{F}_t] \right) \\ &= \text{cl} \left[\left(G_t^M(w) - \cdot \beta_{t,s}(\mathbb{Q}, w) \right) + \left(G_t^M(w) - \cdot \mathbb{E}^{\mathbb{Q}} [\beta_s(\mathbb{Q}, w_t^s(\mathbb{Q}, w)) | \mathcal{F}_t] \right) \right] \\ &= \text{cl} \left[-\bar{\beta}_{t,s}(\mathbb{Q}, w) + \left(\mathbb{E}^{\mathbb{Q}} [G_s^M(w_t^s(\mathbb{Q}, w) - \cdot \beta_s(\mathbb{Q}, w_t^s(\mathbb{Q}, w)) | \mathcal{F}_t] \right) \right] \\ &= \text{cl} \left(-\bar{\beta}_{t,s}(\mathbb{Q}, w) + \mathbb{E}^{\mathbb{Q}} [-\bar{\beta}_s(\mathbb{Q}, w_t^s(\mathbb{Q}, w)) | \mathcal{F}_t] \right). \end{aligned}$$

Again, the third and fourth equations follow from Proposition A.2 and Proposition A.3, respectively. The assumptions of Proposition A.2 are satisfied as $\beta_t(\mathbb{Q}, w) \neq M_t$ for $(\mathbb{Q}, w) \in \mathcal{W}_t$ and thus by the cocycle condition $A := \beta_{t,s}(\mathbb{Q}, w) = \text{cl} [A + G_t^M(w)] \neq M_t$ and $B := \mathbb{E}^{\mathbb{Q}} [\beta_s(\mathbb{Q}, w_t^s(\mathbb{Q}, w)) | \mathcal{F}_t] = \text{cl} [B + G_t^M(w)] \neq M_t$. \square

A.4 Proof of Theorem 2.7

The proof of Theorem 2.7 uses the following two propositions.

Proposition A.4. *Let $w \in M_{t,+}^+ \setminus M_t^\perp$. For any decomposable sets $A, B \in \mathcal{G}(M_t; M_{t,+})$ with $A \neq \emptyset$ if $\mathbb{P}(\text{cl} [B + \Gamma_t^M(w)] = M) > 0$, and $B \neq \emptyset$ if $\mathbb{P}(\text{cl} [A + \Gamma_t^M(w)] = M) > 0$, it holds that*

$$\Gamma_t^M(w) - \cdot \text{cl} [A + B] = \text{cl} [(\Gamma_t^M(w) - \cdot A) + (\Gamma_t^M(w) - \cdot B)]. \quad (\text{A.5})$$

Proof. This follows analogously to the proof of Proposition A.2 noting that, for the second part of the proof, $\text{ess inf}_{b \in B} w^\top b \notin L_t^1(\mathbb{R})$ if and only if either $B = \emptyset$ or $\mathbb{P}(\text{cl} [B + \Gamma_t^M(w)] = M) > 0$. \square

Proposition A.5. *Let $s > t$ and $(\mathbb{Q}, w) \in \mathcal{W}_t^e$. For any decomposable set $A \in \mathcal{G}(M_s; M_{s,+})$ it holds that*

$$\Gamma_t^M(w) - \cdot \mathbb{E}^\mathbb{Q}[A | \mathcal{F}_t] = \text{cl } \mathbb{E}^\mathbb{Q}[\Gamma_s^M(w_t^s(\mathbb{Q}, w)) - \cdot A | \mathcal{F}_t].$$

Proof.

$$\begin{aligned} \text{cl } \mathbb{E}^\mathbb{Q}[\Gamma_s^M(w_t^s(\mathbb{Q}, w)) - \cdot A | \mathcal{F}_t] &= \text{cl} \left\{ \mathbb{E}^\mathbb{Q}[u | \mathcal{F}_t] \mid u \in M_s, u + A \subseteq \Gamma_s^M(w_t^s(\mathbb{Q}, w)) \right\} \\ &= \text{cl} \left\{ \mathbb{E}^\mathbb{Q}[u | \mathcal{F}_t] \mid u \in M_s, \text{ess inf}_{a \in A} w_t^s(\mathbb{Q}, w)^\top(u + a) \geq 0 \right\} \\ &\subseteq \text{cl} \left\{ \mathbb{E}^\mathbb{Q}[u | \mathcal{F}_t] \mid u \in M_s, \mathbb{E} \left[\text{ess inf}_{a \in A} w_t^s(\mathbb{Q}, w)^\top(u + a) \mid \mathcal{F}_t \right] \geq 0 \right\} \\ &= \text{cl} \left\{ \mathbb{E}^\mathbb{Q}[u | \mathcal{F}_t] \mid u \in M_s, \text{ess inf}_{a \in A} w^\top \mathbb{E}^\mathbb{Q}[u + a | \mathcal{F}_t] \geq 0 \right\} \\ &= \text{cl} \left\{ u \in M_t \mid u + \mathbb{E}^\mathbb{Q}[A | \mathcal{F}_t] \subseteq \Gamma_t^M(w) \right\} = \Gamma_t^M(w) - \cdot \mathbb{E}^\mathbb{Q}[A | \mathcal{F}_t]. \end{aligned}$$

We can interchange expectation and essential infimum because A is decomposable and a set of integrable random variables. If $\Gamma_t^M(w) - \cdot \mathbb{E}^\mathbb{Q}[A | \mathcal{F}_t]$ is empty, then equality follows immediately. Now consider a point $u \in \Gamma_t^M(w) - \cdot \mathbb{E}^\mathbb{Q}[A | \mathcal{F}_t]$ and assume $u \notin \text{cl } \mathbb{E}^\mathbb{Q}[\Gamma_s^M(w_t^s(\mathbb{Q}, w)) - \cdot A | \mathcal{F}_t]$. Since $\text{cl } \mathbb{E}^\mathbb{Q}[\Gamma_s^M(w_t^s(\mathbb{Q}, w)) - \cdot A | \mathcal{F}_t]$ is closed and convex, we can separate $\{u\}$ and $\text{cl } \mathbb{E}^\mathbb{Q}[\Gamma_s^M(w_t^s(\mathbb{Q}, w)) - \cdot A | \mathcal{F}_t]$ by some $v \in L_t^q$, i.e.

$$\begin{aligned} \mathbb{E}[v^\top u] &< \inf_{z \in \text{cl } \mathbb{E}^\mathbb{Q}[\Gamma_s^M(w_t^s(\mathbb{Q}, w)) - \cdot A | \mathcal{F}_t]} \mathbb{E}[v^\top z] = \inf_{z \in \Gamma_s^M(w_t^s(\mathbb{Q}, w)) - \cdot A} \mathbb{E}[w_t^s(\mathbb{Q}, v)^\top z] \\ &= \mathbb{E} \left[\text{ess inf}_{z \in \Gamma_s^M(w_t^s(\mathbb{Q}, w)) - \cdot A} w_t^s(\mathbb{Q}, v)^\top z \right]. \end{aligned}$$

Note that in the last equality above we can interchange the expectation and infimum since $\Gamma_s^M(w_t^s(\mathbb{Q}, w)) - \cdot A$ is decomposable and integrable. By construction

$$\text{ess inf}_{z \in \Gamma_s^M(w_t^s(\mathbb{Q}, w)) - \cdot A} w_t^s(\mathbb{Q}, v)^\top z = \begin{cases} \text{ess sup}_{a \in A} (-w_t^s(\mathbb{Q}, w)^\top a) & \text{on } D \\ -\infty & \text{on } D^c, \end{cases}$$

where $D = \{\omega \in \Omega \mid G_0(w_t^s(\mathbb{Q}, v)[\omega]) = G_0(w_t^s(\mathbb{Q}, w)[\omega])\}$. Since $\mathbb{Q} \in \mathcal{M}^e$, one has that $G_0(w_t^s(\mathbb{Q}, v)[\omega]) = G_0(w_t^s(\mathbb{Q}, w)[\omega])$ if and only if $v(\omega) = \lambda(\omega)w(\omega)$ for some $\lambda \in L_t^0(\mathbb{R}_{++})$ (such that $\lambda w \in L_t^q(\mathbb{R})$). Thus, $\mathbb{E} \left[\text{ess inf}_{z \in \Gamma_s^M(w_t^s(\mathbb{Q}, w)) - \cdot A} w_t^s(\mathbb{Q}, v)^\top z \right] > -\infty$ if and only if

$$\begin{aligned} \mathbb{E} \left[\text{ess inf}_{z \in \Gamma_s^M(w_t^s(\mathbb{Q}, w)) - \cdot A} w_t^s(\mathbb{Q}, v)^\top z \right] &= \mathbb{E} \left[\lambda \text{ess inf}_{z \in \Gamma_s^M(w_t^s(\mathbb{Q}, w)) - \cdot A} w_t^s(\mathbb{Q}, w)^\top z \right] \\ &= \mathbb{E} \left[\lambda \text{ess sup}_{a \in A} (-w^\top \mathbb{E}^\mathbb{Q}[a | \mathcal{F}_t]) \right]. \end{aligned}$$

But this implies $\mathbb{E}[\lambda w^\top u] < \mathbb{E}[\lambda \text{ess sup}_{a \in A} (-w^\top \mathbb{E}^\mathbb{Q}[a | \mathcal{F}_t])]$, which is a contradiction to $u \in \Gamma_t^M(w) - \cdot \mathbb{E}^\mathbb{Q}[A | \mathcal{F}_t]$. \square

Proof of Theorem 2.7. Note that $\alpha_t(\mathbb{Q}, w) = \Gamma_t^M(w) - \cdot (-\bar{\alpha}_t(\mathbb{Q}, w))$ and $-\bar{\alpha}_t(\mathbb{Q}, w) = \Gamma_t^M(w) - \cdot \alpha_t(\mathbb{Q}, w)$ for every $(\mathbb{Q}, w) \in \mathcal{W}_t$. The proof is analog to the proof of Theorem 2.6, but using Propositions A.4 and A.5 instead, where the assumptions of Proposition A.4 are satisfied as for $(\mathbb{Q}, w) \in \mathcal{W}_t^e$, $-\bar{\alpha}_t(\mathbb{Q}, w) \neq \emptyset$ and $\mathbb{P}(\alpha_t(\mathbb{Q}, w) = M) = 0$. \square

B Proofs for Section 3

The proof of Lemma 3.2 will need the following dual representation of the scalarization of a set-valued risk measure.

B.1 Set-valued scalarization

Proposition B.1. *Let R_t be a c.u.c. and convex risk measure. Consider $w \in \text{recc}(R_t(0))^+ \setminus M_t^\perp$. Then, for every $X \in L^p$ the following holds*

$$\rho_t(X) := \inf_{u \in R_t(X)} \mathbb{E} [w^\top u] = \sup_{(\mathbb{Q}, m_\perp) \in \mathcal{W}_t(w)} \inf_{Y \in A_t} \mathbb{E} [(w + m_\perp)^\top \mathbb{E}^\mathbb{Q} [Y - X | \mathcal{F}_t]],$$

where $\mathcal{W}_t(w) := \{(\mathbb{Q}, m_\perp) \in \mathcal{M} \times M_t^\perp \mid (\mathbb{Q}, w + m_\perp) \in \mathcal{W}_t\}$.

Proof. Clearly, $\rho_t(X) = \inf_{u \in \text{cl}_M(R_t(X) + G_t^M(w))} \mathbb{E} [w^\top u]$. It holds by a separation argument that

$$\begin{aligned} A_t^w &:= \{X \in L^p \mid \rho_t(X) \leq 0\} \\ &= \{X \in L^p \mid 0 \in \text{cl}_M(R_t(X) + G_t^M(w))\} = \text{cl}_M(A_t + G_t^M(w)). \end{aligned}$$

In fact, $A_t^w = \text{cl}_M(A_t + G_t^M(w + m_\perp))$ for any $m_\perp \in M_t^\perp$. See [36, Definition 2.15] for a discussion of the directional closure cl_M .

The biconjugate is given by $\rho_t^{**}(X) = \sup_{Z \in L^q} (\mathbb{E} [Z^\top X] - \rho_t^*(Z))$, where the convex conjugate is defined by $\rho_t^*(Z) = \sup_{X \in L^p} (\mathbb{E} [Z^\top X] - \rho_t(X))$. Let us calculate the effective domain of ρ_t^* . Consider first $Z \notin L^q$. Let $\hat{Y} \in A_t$ and $Y \in L_+^p$ such that $\mathbb{E} [Z^\top Y] > 0$. In particular, $\hat{Y} + \lambda Y \in A_t$ for any $\lambda > 0$. Therefore,

$$\begin{aligned} \rho_t^*(Z) &= \sup_{X \in L^p} (\mathbb{E} [Z^\top X] - \rho_t(X)) \geq \sup_{\lambda > 0} (\mathbb{E} [Z^\top (\hat{Y} + \lambda Y)] - \rho_t(\hat{Y} + \lambda Y)) \\ &\geq \sup_{\lambda > 0} \mathbb{E} [Z^\top (\hat{Y} + \lambda Y)] = \mathbb{E} [Z^\top \hat{Y}] + \sup_{\lambda > 0} \lambda \mathbb{E} [Z^\top Y] = \infty. \end{aligned}$$

Now consider $\mathbb{E} [Z | \mathcal{F}_t] \notin -w + M_t^\perp$. Let $m \in G_t^M(\mathbb{E} [Z | \mathcal{F}_t]) \cap G_t^M(w)$ such that $\mathbb{E} [Z^\top m] + \mathbb{E} [w^\top m] > 0$. Therefore,

$$\begin{aligned} \rho_t^*(Z) &= \sup_{X \in L^p} (\mathbb{E} [Z^\top X] - \rho_t(X)) \geq \sup_{\lambda > 0} (\mathbb{E} [Z^\top (\lambda m)] - \rho_t(\lambda m)) \\ &= -\rho_t(0) + \sup_{\lambda > 0} \lambda (\mathbb{E} [Z^\top m] + \mathbb{E} [w^\top m]) = \infty. \end{aligned}$$

This implies that $\rho_t^*(Z) \neq \infty$ only if $Z = -w_t^T(\mathbb{Q}, w + m_\perp)$ for some $(\mathbb{Q}, m_\perp) \in \mathcal{W}_t(w)$.

It remains to show that $\rho_t^*(-w_t^T(\mathbb{Q}, w + m_\perp)) = \sup_{Y \in A_t} \mathbb{E} [-(w + m_\perp)^\top \mathbb{E}^\mathbb{Q} [Y | \mathcal{F}_t]]$ for any $(\mathbb{Q}, m_\perp) \in \mathcal{W}_t(w)$. We will prove this by two inequalities. Let $(\mathbb{Q}, m_\perp) \in \mathcal{W}_t(w)$.

$$\begin{aligned} \rho_t^*(-w_t^T(\mathbb{Q}, w + m_\perp)) &= \sup_{X \in L^p} (\mathbb{E} [-(w + m_\perp)^\top \mathbb{E}^\mathbb{Q} [X | \mathcal{F}_t]] - \rho_t(X)) \\ &\geq \sup_{Y \in A_t^w} (\mathbb{E} [-(w + m_\perp)^\top \mathbb{E}^\mathbb{Q} [Y | \mathcal{F}_t]] - \rho_t(Y)) \\ &\geq \sup_{Y \in A_t^w} \mathbb{E} [-(w + m_\perp)^\top \mathbb{E}^\mathbb{Q} [Y | \mathcal{F}_t]] \geq \sup_{Y \in A_t} \mathbb{E} [-(w + m_\perp)^\top \mathbb{E}^\mathbb{Q} [Y | \mathcal{F}_t]]. \end{aligned}$$

For the reverse inequality, denote for those $X \in L^p$ with $R_t(X) \neq \emptyset$ by $u_X \in M_t$ the random variable satisfying $\mathbb{E}[w^\top u_X] = \rho_t(X)$. Then, $X + u_X \in A_t^w$ and

$$\begin{aligned}
\rho_t^*(-w_t^T(\mathbb{Q}, w + m_\perp)) &= \sup_{X \in L^p} \left(\mathbb{E} \left[-(w + m_\perp)^\top \mathbb{E}^\mathbb{Q}[X | \mathcal{F}_t] \right] - \rho_t(X) \right) \\
&= \sup_{\substack{X \in L^p: \\ R_t(X) \neq \emptyset}} \left(\mathbb{E} \left[-(w + m_\perp)^\top \mathbb{E}^\mathbb{Q}[X | \mathcal{F}_t] \right] - \rho_t(X) \right) \\
&= \sup_{\substack{X \in L^p: \\ R_t(X) \neq \emptyset}} \mathbb{E} \left[-(w + m_\perp)^\top \mathbb{E}^\mathbb{Q}[X + u_X | \mathcal{F}_t] \right] \leq \sup_{Y \in A_t^w} \mathbb{E} \left[-(w + m_\perp)^\top \mathbb{E}^\mathbb{Q}[Y | \mathcal{F}_t] \right] \\
&\leq \sup_{Y \in \text{cl}(A_t + G_t^M(w + m_\perp))} \mathbb{E} \left[-(w + m_\perp)^\top \mathbb{E}^\mathbb{Q}[Y | \mathcal{F}_t] \right] = \sup_{Y \in A_t} \mathbb{E} \left[-(w + m_\perp)^\top \mathbb{E}^\mathbb{Q}[Y | \mathcal{F}_t] \right].
\end{aligned} \tag{B.1}$$

Therefore we obtain that

$$\rho_t^{**}(X) = \sup_{(\mathbb{Q}, m_\perp) \in \mathcal{W}_t(w)} \inf_{Y \in A_t} \mathbb{E} \left[(w + m_\perp)^\top \mathbb{E}^\mathbb{Q}[Y - X | \mathcal{F}_t] \right].$$

As ρ_t is proper (by Corollary B.6), convex by convexity of R_t , and lower semicontinuous (by Proposition B.3 since $\text{recc}(R_t(0))^+ \subseteq M_{t,+}^+$), the Fenchel-Moreau Theorem yields the assertion. \square

Corollary B.2. *Let $R_{t,s}$ be a c.u.c. and convex stepped risk measure. Consider $w \in \text{recc}(R_t(0))^+ \setminus M_t^\perp$. Then, for every $X \in M_s$ the following holds*

$$\inf_{u \in R_t(X)} \mathbb{E} \left[w^\top u \right] = \sup_{(\mathbb{Q}, m_\perp) \in \mathcal{W}_{t,s}(w)} \inf_{Y \in A_{t,s}} \mathbb{E} \left[(w + m_\perp)^\top \mathbb{E}^\mathbb{Q}[Y - X | \mathcal{F}_t] \right],$$

where

$$\begin{aligned}
\mathcal{W}_{t,s}(w) &:= \left\{ (\mathbb{Q}, m_\perp) \in \mathcal{M} \times M_t^\perp \mid (\mathbb{Q}, w + m_\perp) \in \mathcal{W}_{t,s} \right\} \supseteq \mathcal{W}_t(w) \\
\mathcal{W}_{t,s} &= \left\{ (\mathbb{Q}, v) \in \mathcal{M} \times \left(M_{t,+}^+ \setminus M_t^\perp \right) \mid w_t^s(\mathbb{Q}, v) \in M_{s,+}^+, \mathbb{Q} = \mathbb{P}|_{\mathcal{F}_t} \right\}.
\end{aligned}$$

Proof. This follows similarly to Proposition B.1. \square

The following proposition is a modification of [40, Proposition 3.29] by noting that the only open sets needed in that proof are those with a closed and convex complement. However, for the convenience of the reader, a proof is provided here as well.

Proposition B.3. *If R_t is c.u.c., the scalarization $\rho_t(X) := \inf_{u \in R_t(X)} \mathbb{E}[w^\top u]$ is lower semicontinuous for any $w \in M_{t,+}^+ \setminus M_t^\perp$.*

Proof. By definition, R_t is c.u.c. if and only if $R_t^+(V) := \{X \in L^p \mid R_t(X) \subseteq V\}$ is open for any $V \subseteq M_t$ such that the complement V^c is closed and convex. For any $\epsilon > 0$, $V_\epsilon = \{m \in M_t \mid \mathbb{E}[w^\top m] > -\epsilon\}$ is an open neighborhood of 0. Thus, $R_t(X_0) + V_\epsilon$ is open for any X_0 and the complement is convex via:

$$\begin{aligned}
[R_t(X_0) + V_\epsilon]^c &= \left\{ m \in M_t \mid \mathbb{E} \left[w^\top (m - u) \right] \leq -\epsilon \ \forall u \in R_t(X_0) \right\} \\
&= \left\{ m \in M_t \mid \mathbb{E} \left[w^\top m \right] \leq -\epsilon + \inf_{u \in R_t(X)} \mathbb{E} \left[w^\top u \right] \right\}.
\end{aligned}$$

Therefore $R_t^+(R_t(X_0) + V_\epsilon)$ is open, and trivially is a neighborhood of X_0 . Thus, $\rho_t(X) \geq \rho_t(X_0) - \epsilon$ for any $X \in R_t^+(R_t(X_0) + V_\epsilon)$, which implies lower semicontinuity at X_0 . Since this is true for any X_0 , the result is proven. \square

Proposition B.4. *Let R_t be a closed and convex risk measure. The scalarization ρ_t is proper if and only if $w \in \bigcap_{\substack{X \in L^p \\ R_t(X) \neq \emptyset}} \text{recc}(R_t(X))^+$.*

Proof. Clearly, $\rho_t(0) < \infty$ for any $w \in M_{t,+}^+$ since $R_t(0) \neq \emptyset$. And for any $X \in L^p$ with $R_t(X) \neq \emptyset$, we know $\rho_t(X) > -\infty$ if and only if $w \in \text{recc}(R_t(X))^+$ by the definition of the recession cone. For $X \in L^p$ with $R_t(X) = \emptyset$, $\rho_t(X) > -\infty$ is trivially true. \square

Corollary B.5. *If R_t is a closed convex risk measure and $X \in L^p$ with $R_t(X) \neq \emptyset$. Then, $\text{recc}(R_t(X)) = \bigcap_{(\mathbb{Q}, w) \in \mathcal{W}_t^t} G_t^M(w)$.*

Proof. $r \in \text{recc}(R_t(X))$ if and only if $r \in \bigcap_{\lambda > 0} \lambda(R_t(X) - u)$ for any $u \in R_t(X)$ if and only if $u + \frac{1}{\lambda}r \in R_t(X)$ for any $\lambda > 0$ and any $u \in R_t(X)$. Using (2.1) and noting that one can replace \mathcal{W}_t with \mathcal{W}_t^t , this is equivalent to $\mathbb{E}[w^\top(u + \frac{1}{\lambda}r)] \geq \inf_{Y \in A_t} \mathbb{E}[w^\top \mathbb{E}^\mathbb{Q}[Y - X | \mathcal{F}_t]]$ for every $(\mathbb{Q}, w) \in \mathcal{W}_t^t$, every $\lambda > 0$ and every $u \in R_t(X)$. This is true if and only if for every $(\mathbb{Q}, w) \in \mathcal{W}_t^t$ and $u \in R_t(X)$

$$\mathbb{E}[w^\top r] \geq \sup_{\lambda > 0} \lambda \left[\inf_{Y \in A_t} \mathbb{E}[w^\top \mathbb{E}^\mathbb{Q}[Y - X | \mathcal{F}_t]] - \mathbb{E}[w^\top u] \right] = 0,$$

where the last equality holds since $u \in R_t(X)$. This yields the assertion. \square

Corollary B.6. *If R_t is a closed convex risk measure, then ρ_t is proper if and only if $w \in \text{recc}(R_t(0))^+ = (\bigcap_{(\mathbb{Q}, w) \in \mathcal{W}_t^t} G_t^M(w))^+$.*

Proof. This follows from Proposition B.4 and Corollary B.5. \square

B.2 Proof of Lemma 3.2

Throughout we will use the following set of dual variables for times $t \leq s \leq T$

$$\mathcal{W}_t^s := \{(\mathbb{Q}, w) \in \mathcal{W}_t \mid \beta_s(\mathbb{Q}, w_t^s(\mathbb{Q}, w)) \neq \emptyset\}.$$

Proof. “ \Leftarrow ” We will prove first that conditions (3.2) and (3.3) imply the supermartingale property (3.1). If $(\mathbb{Q}, w) \notin \mathcal{W}_t^t$, then $V_t^{(\mathbb{Q}, w)}(X) = \emptyset$ for any $X \in L^p$ and (3.1) is satisfied. Now, let $X \in L^p$ and $(\mathbb{Q}, w) \in \mathcal{W}_t^t$. It holds

$$V_t^{(\mathbb{Q}, w)}(X) = \left\{ u \in M_t \mid \mathbb{E}[w^\top u] \geq \inf_{Z \in R_t(X)} \mathbb{E}[w^\top Z] + \sup_{Y_t \in A_t} \mathbb{E}[w^\top \mathbb{E}^\mathbb{Q}[-Y_t | \mathcal{F}_t]] \right\}.$$

Similarly, it follows that

$$\begin{aligned} & \mathbb{E}^\mathbb{Q} \left[V_s^{(\mathbb{Q}, w_t^s(\mathbb{Q}, w))}(X) \mid \mathcal{F}_t \right] \\ &= \left\{ u \in M_t \mid \mathbb{E}[w^\top u] \geq \inf_{Z \in R_s(X)} \mathbb{E}[w^\top \mathbb{E}^\mathbb{Q}[Z | \mathcal{F}_t]] + \sup_{Y_s \in A_s} \mathbb{E}[w^\top \mathbb{E}^\mathbb{Q}[-Y_s | \mathcal{F}_t]] \right\} \end{aligned}$$

if $\beta_s(\mathbb{Q}, w_t^s(\mathbb{Q}, w)) \neq \emptyset$, i.e., if $(\mathbb{Q}, w) \in \mathcal{W}_t^s$. The latter is true as we will now show that condition (3.2) yields in fact $\mathcal{W}_t^t \subseteq \mathcal{W}_t^s$. Note that condition (3.2) implies $A_t \supseteq A_s + A_{t,s}$ (Lemma 3.6(iii) in [20]) which yields for every $(\mathbb{Q}, w) \in \mathcal{W}_t$

$$\beta_t(\mathbb{Q}, w) \subseteq \text{cl} \left(\beta_{t,s}(\mathbb{Q}, w) + \mathbb{E}^{\mathbb{Q}} [\beta_s(\mathbb{Q}, w_t^s(\mathbb{Q}, w)) | \mathcal{F}_t] \right).$$

This follows from a trivial modification to the first part of the proof of Theorem 3.2 in [22], followed by switching to the positive conjugate via $\beta_t(\mathbb{Q}, w) = G_t^M(w) - (-\bar{\beta}_t(\mathbb{Q}, w))$ and using Propositions A.2 and A.3. Therefore, if $\beta_t(\mathbb{Q}, w) \neq \emptyset$ it must follow that $\beta_s(\mathbb{Q}, w_t^s(\mathbb{Q}, w)) \neq \emptyset$, i.e. $\mathcal{W}_t^t \subseteq \mathcal{W}_t^s$. Thus, the supermartingale property holds via

$$\begin{aligned} V_t^{(\mathbb{Q}, w)}(X) &= \left\{ u \in M_t \mid \mathbb{E} [w^\top u] \geq \inf_{Z_t \in R_t(X)} \sup_{Y_t \in A_t} \mathbb{E} [w^\top (Z_t - \mathbb{E}^{\mathbb{Q}} [Y_t | \mathcal{F}_t])] \right\} \\ &\subseteq \left\{ u \in M_t \mid \mathbb{E} [w^\top u] \geq \inf_{Z_t \in R_t(X)} \sup_{\substack{Y_{t,s} \in A_{t,s} \\ Y_s \in A_s}} \mathbb{E} [w^\top (Z_t - \mathbb{E}^{\mathbb{Q}} [Y_{t,s} + Y_s | \mathcal{F}_t])] \right\} \quad (\text{B.2}) \\ &= \left\{ u \in M_t \mid \mathbb{E} [w^\top u] \geq \inf_{Z_t \in R_t(X)} \sup_{Y_{t,s} \in A_{t,s}} \mathbb{E} [w^\top (Z_t - \mathbb{E}^{\mathbb{Q}} [Y_{t,s} | \mathcal{F}_t])] \right\} \\ &\quad + \left\{ u \in M_t \mid \mathbb{E} [w^\top u] \geq \sup_{Y_s \in A_s} \mathbb{E} [-w^\top \mathbb{E}^{\mathbb{Q}} [Y_s | \mathcal{F}_t]] \right\} \\ &\subseteq \left\{ u \in M_t \mid \mathbb{E} [w^\top u] \geq \inf_{Z_s \in R_s(X)} \mathbb{E} [w^\top \mathbb{E}^{\mathbb{Q}} [Z_s | \mathcal{F}_t]] \right\} \\ &\quad + \left\{ u \in M_t \mid \mathbb{E} [w^\top u] \geq \sup_{Y_s \in A_s} \mathbb{E} [-w^\top \mathbb{E}^{\mathbb{Q}} [Y_s | \mathcal{F}_t]] \right\} \quad (\text{B.3}) \\ &= \left\{ u \in M_t \mid \mathbb{E} [w^\top u] \geq \inf_{Z_s \in R_s(X)} \sup_{Y_s \in A_s} \mathbb{E} [w^\top \mathbb{E}^{\mathbb{Q}} [Z_s - Y_s | \mathcal{F}_t]] \right\} \\ &= \mathbb{E}^{\mathbb{Q}} [V_s^{(\mathbb{Q}, w_t^s(\mathbb{Q}, w))}(X) | \mathcal{F}_t]. \end{aligned}$$

Inclusion (B.2) follows from condition (3.2) (as (3.2) implies $A_t \supseteq A_s + A_{t,s}$ by Lemma 3.6(iii) in [20]). Inclusion (B.3) is true if and only if

$$\inf_{Z_t \in R_t(X)} \mathbb{E} [w^\top Z_t] \geq \inf_{Z_s \in R_s(X)} \inf_{Y_{t,s} \in A_{t,s}} \mathbb{E} [w^\top \mathbb{E}^{\mathbb{Q}} [Z_s + Y_{t,s} | \mathcal{F}_t]].$$

But this follows from $R_t(X) \subseteq \text{cl} \bigcup_{Z \in R_s(X)} ((\mathbb{E}^{\mathbb{Q}} [Z | \mathcal{F}_t] + G_t(w)) \cap M_t - \beta_{t,s}(\mathbb{Q}, w))$, which is immediate by condition (3.3).

“ \Rightarrow ” We will now prove that the supermartingale property (3.1) implies (3.2) and (3.3). First note that (3.1) yields $\mathcal{W}_t^t \subseteq \mathcal{W}_t^s$: Assume $\beta_s(\mathbb{Q}, w_t^s(\mathbb{Q}, w)) = \emptyset$, then $V_s^{(\mathbb{Q}, w_t^s(\mathbb{Q}, w))}(0) = \emptyset$ by definition of the Minkowski addition. This implies $V_t^{(\mathbb{Q}, w)}(0) = \emptyset$ by the supermartingale relation. However this can only occur if $\beta_t(\mathbb{Q}, w) = \emptyset$ since $R_t(0) \neq \emptyset$ by definition.

We will now show that the supermartingale property (3.1) implies time consistency, which then yields condition (3.2). Assume $R_s(X) \subseteq R_s(Y)$ for some $X, Y \in L^p$. We need to show that $R_t(X) \subseteq R_t(Y)$. Let $(\mathbb{Q}, w) \in \mathcal{W}_t^t$. By $\mathcal{W}_t^t \subseteq \mathcal{W}_t^s$, it follows that

$\beta_s(\mathbb{Q}, w_t^s(\mathbb{Q}, w)) \neq \emptyset$. Also assume $R_t(X) \neq \emptyset$ (if it were empty then it would trivially follow that $R_t(X) \subseteq R_t(Y)$). It holds

$$\begin{aligned}
\text{cl}[R_t(X) + \beta_t(\mathbb{Q}, w)] &= V_t^{(\mathbb{Q}, w)}(X) \subseteq \mathbb{E}^\mathbb{Q} \left[V_s^{(\mathbb{Q}, w_t^s(\mathbb{Q}, w))}(X) \middle| \mathcal{F}_t \right] \\
&= \mathbb{E}^\mathbb{Q} [\text{cl}[R_s(X) + \beta_s(\mathbb{Q}, w_t^s(\mathbb{Q}, w))] | \mathcal{F}_t] \subseteq \mathbb{E}^\mathbb{Q} [\text{cl}[R_s(Y) + \beta_s(\mathbb{Q}, w_t^s(\mathbb{Q}, w))] | \mathcal{F}_t] \\
&\subseteq \mathbb{E}^\mathbb{Q} \left[\text{cl} \left[\left((\mathbb{E}^\mathbb{Q}[-Y | \mathcal{F}_s] + G_s(w_t^s(\mathbb{Q}, w))) \cap M_s - \cdot \beta_s(\mathbb{Q}, w_t^s(\mathbb{Q}, w)) \right) \right. \right. \\
&\quad \left. \left. + \beta_s(\mathbb{Q}, w_t^s(\mathbb{Q}, w)) \right) \middle| \mathcal{F}_t \right] \\
&\subseteq \mathbb{E}^\mathbb{Q} \left[(\mathbb{E}^\mathbb{Q}[-Y | \mathcal{F}_s] + G_s(w_t^s(\mathbb{Q}, w))) \cap M_s \middle| \mathcal{F}_t \right] = (\mathbb{E}^\mathbb{Q}[-Y | \mathcal{F}_t] + G_t(w)) \cap M_t.
\end{aligned}$$

The last equality follows from the tower property and [22, Corollary A.4]. The last inclusion follows from $\text{cl}[(A + G_s^M] - \cdot B) + B] \subseteq \text{cl}[A + G_s^M]$, which holds by the definition of the Minkowski subtraction. Here, we have $A = (\mathbb{E}^\mathbb{Q}[-Y | \mathcal{F}_s] + G_s(w_t^s(\mathbb{Q}, w))) \cap M_s$, $B = \beta_s(\mathbb{Q}, w_t^s(\mathbb{Q}, w))$, and notice that $\text{cl}[A + G_s^M] = A$, where we used the notation $G_s^M = G_s^M(w_t^s(\mathbb{Q}, w))$.

From the above we have

$$R_t(X) + \beta_t(\mathbb{Q}, w) \subseteq \text{cl}[R_t(X) + \beta_t(\mathbb{Q}, w)] \subseteq (\mathbb{E}^\mathbb{Q}[-Y | \mathcal{F}_t] + G_t(w)) \cap M_t, \quad (\text{B.4})$$

which yields

$$R_t(X) \subseteq R_t(X) + G_t^M(w) \subseteq (R_t(X) + \beta_t(\mathbb{Q}, w)) - \cdot \beta_t(\mathbb{Q}, w) \quad (\text{B.5})$$

$$\subseteq (\mathbb{E}^\mathbb{Q}[-Y | \mathcal{F}_t] + G_t(w)) \cap M_t - \cdot \beta_t(\mathbb{Q}, w) \quad (\text{B.6})$$

for any $(\mathbb{Q}, w) \in \mathcal{W}_t$ (as it trivially holds for those $(\mathbb{Q}, w) \in \mathcal{W}_t$ for which $\beta_t(\mathbb{Q}, w) = \emptyset$ as well). The second inclusion in (B.5) follows from $A + G_t^M(w) \subseteq (A + B + G_t^M(w)) - \cdot B$, which holds by the definition of the Minkowski subtraction. Here, $A = R_t(X)$, $B = \beta_t(\mathbb{Q}, w)$, and $B + G_t^M(w) = B$.

The inclusions (B.5), (B.6) yield

$$R_t(X) \subseteq \bigcap_{(\mathbb{Q}, w) \in \mathcal{W}_t} \left[(\mathbb{E}^\mathbb{Q}[-Y | \mathcal{F}_t] + G_t(w)) \cap M_t - \cdot \beta_t(\mathbb{Q}, w) \right] = R_t(Y).$$

This is time consistency which implies (3.2) as follows. Let $Y \in \bigcup_{Z \in R_s(X)} R_t(-Z)$, i.e. there exists a $\hat{Z} \in R_s(X)$ such that $Y \in R_t(-\hat{Z})$. We need to show that $Y \in R_t(X)$. By translitivity and normalization of the risk measure, it holds that

$$R_s(-\hat{Z}) = R_s(0) + \hat{Z} \subseteq R_s(0) + R_s(X) = R_s(X).$$

Time consistency now yields $R_t(-\hat{Z}) \subseteq R_t(X)$ and as $Y \in R_t(-\hat{Z})$, it holds that $Y \in R_t(X)$ and thus (3.2).

We will now prove that the supermartingale property (3.1) implies (3.3). Let $(\mathbb{Q}, w) \in \mathcal{W}_t^t$. By (B.5),

$$\begin{aligned}
R_t(X) &\subseteq (R_t(X) + \beta_t(\mathbb{Q}, w)) - \cdot \beta_t(\mathbb{Q}, w) \subseteq \text{cl}[R_t(X) + \beta_t(\mathbb{Q}, w)] - \cdot \beta_t(\mathbb{Q}, w) \\
&= V_t^{(\mathbb{Q}, w)}(X) - \cdot \beta_t(\mathbb{Q}, w) \subseteq \mathbb{E}^\mathbb{Q} \left[V_s^{(\mathbb{Q}, w_t^s(\mathbb{Q}, w))}(X) \middle| \mathcal{F}_t \right] - \cdot \beta_t(\mathbb{Q}, w) \\
&= \mathbb{E}^\mathbb{Q} [\text{cl}[R_s(X) + \beta_s(\mathbb{Q}, w_t^s(\mathbb{Q}, w))] | \mathcal{F}_t] - \cdot \beta_t(\mathbb{Q}, w) \\
&= \mathbb{E}^\mathbb{Q} \left[\text{cl} \left(\bigcup_{Z \in R_s(X)} [R_s(-Z) + \beta_s(\mathbb{Q}, w_t^s(\mathbb{Q}, w))] \right) \middle| \mathcal{F}_t \right] - \cdot \beta_t(\mathbb{Q}, w) \\
&\subseteq \text{cl} \left(\bigcup_{Z \in R_s(X)} \left[\mathbb{E}^\mathbb{Q} [R_s(-Z) + \beta_s(\mathbb{Q}, w_t^s(\mathbb{Q}, w))] | \mathcal{F}_t \right] - \cdot \beta_t(\mathbb{Q}, w) \right) \\
&\subseteq \text{cl} \left(\bigcup_{Z \in R_s(X)} \left[\mathbb{E}^\mathbb{Q} \left[(\mathbb{E}^\mathbb{Q} [Z | \mathcal{F}_s] + G_s(w_t^s(\mathbb{Q}, w))) \cap M_s \middle| \mathcal{F}_t \right] - \cdot \beta_t(\mathbb{Q}, w) \right] \right) \quad (\text{B.7}) \\
&= \text{cl} \left(\bigcup_{Z \in R_s(X)} \left[\mathbb{E}^\mathbb{Q} [Z | \mathcal{F}_t] + G_t(w) \cap M_t - \cdot \beta_t(\mathbb{Q}, w) \right] \right).
\end{aligned}$$

Inclusion (B.7) holds as for any $\tilde{s} > s$, $R_{\tilde{s}}(-Z) \subseteq R_s(-Z)$, which yields by (B.4) (setting $X = Y = -Z$) that

$$R_s(-Z) + \beta_s(\mathbb{Q}, w_t^s(\mathbb{Q}, w)) \subseteq (\mathbb{E}^\mathbb{Q} [Z | \mathcal{F}_s] + G_s(w_t^s(\mathbb{Q}, w))) \cap M_s.$$

The last equality follows from the tower property and [22, Corollary A.4].

From the chain of inclusions above, we therefore conclude that for all dual variables in $(\mathbb{Q}, w) \in \mathcal{W}_t$ (as it trivially holds also for those $(\mathbb{Q}, w) \in \mathcal{W}_t$ that are not in \mathcal{W}_t^t)

$$R_t(X) \subseteq \bigcap_{(\mathbb{Q}, w) \in \mathcal{W}_t} \text{cl} \bigcup_{Z \in R_s(X)} \left[(\mathbb{E}^\mathbb{Q} [Z | \mathcal{F}_t] + G_t(w)) \cap M_t - \cdot \beta_t(\mathbb{Q}, w) \right]. \quad (\text{B.8})$$

To prove (3.3), it remains to show that we can replace $\beta_t(\mathbb{Q}, w)$ in (B.8) by the stepped version $\beta_{t,s}(\mathbb{Q}, w)$. Let $Z \in M_s$. And let $\rho_{t,s}(Z) := \inf_{u \in R_{t,s}(Z)} \mathbb{E} [w^\top u]$ for $w \in \text{recc}(R_t(0))^+ \setminus M_t^\perp$, which is a proper, convex, lower semicontinuous function (see Proposition B.3 and Corollary B.6) with representation given in Corollary B.2. Note that the first and last lines below follow from $\rho_t(Z) = -\infty$ for $w \notin \text{recc}(R_t(0))^+$ by Corollary B.6. Also, note that $R_{t,s}(0) = R_t(0)$ by definition. The representation in the first and last lines follow from a separation argument.

$$\begin{aligned}
R_{t,s}(Z) &= \bigcap_{w \in \text{recc}(R_t(0))^+ \setminus M_t^\perp} \left\{ m \in M_t \mid \mathbb{E} \left[w^\top m \right] \geq \rho_{t,s}(Z) \right\} \\
&= \bigcap_{w \in \text{recc}(R_t(0))^+ \setminus M_t^\perp} \left\{ m \in M_t \mid \mathbb{E} \left[w^\top m \right] \geq \right. \\
&\quad \left. \sup_{(\mathbb{Q}, m_\perp) \in \mathcal{W}_{t,s}(w)} \inf_{Y_{t,s} \in A_{t,s}} \mathbb{E} \left[(w + m_\perp)^\top \mathbb{E}^\mathbb{Q} [Y_{t,s} - Z \mid \mathcal{F}_t] \right] \right\} \\
&\subseteq \bigcap_{w \in \text{recc}(R_t(0))^+ \setminus M_t^\perp} \left\{ m \in M_t \mid \mathbb{E} \left[w^\top m \right] \geq \right. \\
&\quad \left. \sup_{(\mathbb{Q}, m_\perp) \in \mathcal{W}_t(w)} \inf_{Y_t \in A_t} \mathbb{E} \left[(w + m_\perp)^\top \mathbb{E}^\mathbb{Q} [Y_t - Z \mid \mathcal{F}_t] \right] \right\} \\
&= \bigcap_{w \in \text{recc}(R_t(0))^+ \setminus M_t^\perp} \left\{ m \in M_t \mid \mathbb{E} \left[w^\top m \right] \geq \rho_t(Z) \right\} \\
&= R_t(Z) = R_{t,s}(Z).
\end{aligned}$$

Therefore, for every $Z \in M_s$ and every $w \in \text{recc}(R_t(0))^+ \setminus M_t^\perp$ it holds that for all $(\mathbb{Q}, m_\perp) \in \mathcal{W}_t(w)$ there exists an $(\mathbb{R}, n_\perp) \in \mathcal{W}_t(w)$ such that

$$\inf_{Y_{t,s} \in A_{t,s}} \mathbb{E} \left[(w + m_\perp)^\top \mathbb{E}^\mathbb{Q} [Y_{t,s} - Z \mid \mathcal{F}_t] \right] \leq \inf_{Y_t \in A_t} \mathbb{E} \left[(w + n_\perp)^\top \mathbb{E}^\mathbb{R} [Y_t - Z \mid \mathcal{F}_t] \right].$$

This is because every such constraint is “active”, i.e., if any were made any stricter it would shrink the set $R_{t,s}(Z)$. In particular, this is true if $m_\perp = 0 \in M_t^\perp$. Additionally, for $w \notin \text{recc}(R_t(0))^+ \setminus M_t^\perp$, $\inf_{Y_{t,s} \in A_{t,s}} \mathbb{E} \left[(w + m_\perp)^\top \mathbb{E}^\mathbb{Q} [Y_{t,s} - Z \mid \mathcal{F}_t] \right] = -\infty$ for any $(\mathbb{Q}, m_\perp) \in \mathcal{W}_t(w)$. Thus, we can conclude for every $Z \in M_s$ it holds that for all $(\mathbb{Q}, w) \in \mathcal{W}_t$ there exists an $(\mathbb{R}, v) \in \mathcal{W}_t$ such that $v \in w + M_t^\perp$ and

$$\inf_{Y_{t,s} \in A_{t,s}} \mathbb{E} \left[w^\top \mathbb{E}^\mathbb{Q} [Y_{t,s} - Z \mid \mathcal{F}_t] \right] \leq \inf_{Y_t \in A_t} \mathbb{E} \left[v^\top \mathbb{E}^\mathbb{R} [Y_t - Z \mid \mathcal{F}_t] \right].$$

In particular, for every $(\mathbb{Q}, w) \in \mathcal{W}_t$ there exists some $\mathbb{R}(\mathbb{Q}, w, Z) \in \mathcal{M}$ and $v(\mathbb{Q}, w, Z) \in w + M_t^\perp$ so that $(\mathbb{R}(\mathbb{Q}, w, Z), v(\mathbb{Q}, w, Z)) \in \mathcal{W}_t$ and

$$\begin{aligned}
&\left(\mathbb{E}^\mathbb{Q} [-Z \mid \mathcal{F}_t] + G_t(w) \right) \cap M_t - \cdot \beta_{t,s}(\mathbb{Q}, w) \supseteq \\
&\left(\mathbb{E}^{\mathbb{R}(\mathbb{Q}, w, Z)} [-Z \mid \mathcal{F}_t] + G_t(v(\mathbb{Q}, w, Z)) \right) \cap M_t - \cdot \beta_t(\mathbb{R}(\mathbb{Q}, w, Z), v(\mathbb{Q}, w, Z)).
\end{aligned}$$

Using (B.8), this implies (3.3):

$$\begin{aligned}
R_t(X) &\subseteq \bigcap_{(\mathbb{Q}, w) \in \mathcal{W}_t} \text{cl} \bigcup_{Z \in R_s(X)} \left[\left(\mathbb{E}^\mathbb{Q} [Z \mid \mathcal{F}_t] + G_t(w) \right) \cap M_t - \cdot \beta_t(\mathbb{Q}, w) \right] \\
&\subseteq \bigcap_{(\mathbb{Q}, w) \in \mathcal{W}_t} \text{cl} \bigcup_{Z \in R_s(X)} \left[\left(\mathbb{E}^{\mathbb{R}(\mathbb{Q}, w, -Z)} [Z \mid \mathcal{F}_t] + G_t(v(\mathbb{Q}, w, -Z)) \right) \cap M_t \right. \\
&\quad \left. - \cdot \beta_t(\mathbb{R}(\mathbb{Q}, w, -Z), v(\mathbb{Q}, w, -Z)) \right] \\
&\subseteq \bigcap_{(\mathbb{Q}, w) \in \mathcal{W}_t} \text{cl} \bigcup_{Z \in R_s(X)} \left[\left(\mathbb{E}^\mathbb{Q} [Z \mid \mathcal{F}_t] + G_t(w) \right) \cap M_t - \cdot \beta_{t,s}(\mathbb{Q}, w) \right].
\end{aligned}$$

□

B.3 Proof of Lemma 3.3

Proof. First, note that inclusion (3.2) is equivalent to inclusion (3.4) by [20, Lemma 3.6(iii)]. Second, we will show that inclusion (3.5) is equivalent to

$$R_t(X) \subseteq \bigcap_{(\mathbb{Q}, w) \in \mathcal{W}_t} \bigcup_{Z \in R_s(X)} \left[\left(\mathbb{E}^{\mathbb{Q}}[Z | \mathcal{F}_t] + G_t(w) \right) \cap M_t - \cdot \beta_{t,s}(\mathbb{Q}, w) \right] \quad (\text{B.9})$$

for all $X \in L^p$. Let $(\mathbb{Q}, w) \in \mathcal{W}_t$. To do this, we first show that (B.9) implies (3.5): Let $X \in A_t$. By assumption,

$$\begin{aligned} 0 &\in \bigcup_{Z \in R_s(X)} \left[\left(\mathbb{E}^{\mathbb{Q}}[Z | \mathcal{F}_t] + G_t(w) \right) \cap M_t - \cdot \beta_{t,s}(\mathbb{Q}, w) \right] \\ &= \left\{ m \in M_t \mid \exists Z \in R_s(X) : \mathbb{E} \left[w^\top m \right] \geq \mathbb{E} \left[w^\top \mathbb{E}^{\mathbb{Q}}[Z | \mathcal{F}_t] \right] + \inf_{Y \in A_{t,s}} \mathbb{E} \left[w^\top \mathbb{E}^{\mathbb{Q}}[Y | \mathcal{F}_t] \right] \right\}. \end{aligned}$$

This is true if and only if there exists a $Z \in R_s(X)$ such that

$$\mathbb{E} \left[w^\top \mathbb{E}^{\mathbb{Q}}[-Z | \mathcal{F}_t] \right] \geq \inf_{Y \in A_{t,s}} \mathbb{E} \left[w^\top \mathbb{E}^{\mathbb{Q}}[Y | \mathcal{F}_t] \right],$$

i.e., $-R_s(X) \cap \text{cl} \left(A_{t,s} + G_s^M(w_t^s(\mathbb{Q}, w)) \right) \neq \emptyset$. By [20, Lemma 3.6(i)] this is true if and only if $X \in A_s + \text{cl} \left(A_{t,s} + G_s^M(w_t^s(\mathbb{Q}, w)) \right)$.

Now, we show that (3.5) implies (B.9): Let $m \in R_t(X)$. Then, $X + m \in A_s + \text{cl} \left(A_{t,s} + G_s^M(w_t^s(\mathbb{Q}, w)) \right)$ by assumption. By the equivalences shown above, and transitivity,

$$m \in \bigcup_{Z \in R_s(X)} \left[\left(\mathbb{E}^{\mathbb{Q}}[Z | \mathcal{F}_t] + G_t(w) \right) \cap M_t - \cdot \beta_{t,s}(\mathbb{Q}, w) \right].$$

The last step of the proof is to show that one can remove the closure in (3.3), which reduces then to (B.9). This can be done using that R_s is c.u.c. and convex. For notation let $\hat{R}_{t,s}^{(\mathbb{Q}, w)}(Z) := \left(\mathbb{E}^{\mathbb{Q}}[-Z | \mathcal{F}_t] + G_t(w) \right) \cap M_t - \cdot \beta_{t,s}(\mathbb{Q}, w)$, which is a convex risk measure for any $(\mathbb{Q}, w) \in \mathcal{W}_t$. Then $\tilde{R}_t(X) := \bigcup_{Z \in R_s(X)} \hat{R}_{t,s}^{(\mathbb{Q}, w)}(-Z)$ is closed (for any choice $(\mathbb{Q}, w) \in \mathcal{W}_t$) if $\tilde{A}_t := \left\{ X \in L^p \mid 0 \in \tilde{R}_t(X) \right\}$ is closed. Using the proof of [22, Lemma B.2], $\tilde{A}_t = \tilde{R}_t^{-1}[M_t, -]$, which is closed if $\hat{R}_{t,s}^{(\mathbb{Q}, w), -1}[M_t, -]$ is a closed, convex, upper set (as R_s is c.u.c. and convex). But this is true as

$$\begin{aligned} \hat{R}_{t,s}^{(\mathbb{Q}, w), -1}[M_t, -] &= \left\{ Z \in M_s \mid 0 \geq \inf_{Y \in A_{t,s}} \mathbb{E} \left[w^\top \mathbb{E}^{\mathbb{Q}}[Y | \mathcal{F}_t] \right] + \mathbb{E} \left[w^\top \mathbb{E}^{\mathbb{Q}}[-Z | \mathcal{F}_t] \right] \right\} \\ &= \left\{ Z \in M_s \mid \mathbb{E} \left[w_t^s(\mathbb{Q}, w)^\top Z \right] \geq \inf_{Y \in A_{t,s}} \mathbb{E} \left[w_t^s(\mathbb{Q}, w)^\top Y \right] \right\} \\ &= \text{cl} \left(A_{t,s} + G_s^M(w_t^s(\mathbb{Q}, w)) \right), \end{aligned}$$

which is a closed, convex, and upper set. \square

B.4 Proof of Corollary 3.5

Proof. Fix $X \in L^p$ and let $(\mathbb{Q}, w) \in \mathcal{W}_0$ such that $\beta_0(\mathbb{Q}, w) \neq \emptyset$ and

$$\text{cl} \left[R_0(X) + G_0^M(w) \right] = \left(\mathbb{E}^{\mathbb{Q}}[-X] + G_0(w) \right) \cap M_0 - \cdot \beta_0(\mathbb{Q}, w).$$

Define $m_t^X \in M_t$ such that

$$m_t^X + G_t^M(w_0^t(\mathbb{Q}, w)) = \left(\mathbb{E}^\mathbb{Q}[X | \mathcal{F}_t] + G_t(w_0^t(\mathbb{Q}, w)) \right) \cap M_t$$

and let $U_t := V_t^{(\mathbb{Q}, w_0^t(\mathbb{Q}, w))}(X) + m_t^X$ for any time t . We claim that $U_t = G_t^M(w_0^t(\mathbb{Q}, w))$ for any time t , which we will then use to prove that $V_t^{(\mathbb{Q}, w_0^t(\mathbb{Q}, w))}(X)$ is a \mathbb{Q} -martingale. Let us first show that $U_t \supseteq G_t^M(w_0^t(\mathbb{Q}, w))$:

$$\begin{aligned} \inf_{u \in U_t} \mathbb{E} \left[w_0^t(\mathbb{Q}, w)^\top u \right] &= \inf_{v \in V_t^{(\mathbb{Q}, w_0^t(\mathbb{Q}, w))}(X)} \mathbb{E} \left[w_0^t(\mathbb{Q}, w)^\top v \right] + \mathbb{E} \left[w_0^t(\mathbb{Q}, w)^\top m_t^X \right] \\ &= \inf_{v \in V_t^{(\mathbb{Q}, w_0^t(\mathbb{Q}, w))}(X)} w^\top \mathbb{E}^\mathbb{Q} [v] + w^\top \mathbb{E}^\mathbb{Q} [X] \leq \inf_{v \in V_0^{(\mathbb{Q}, w)}} w^\top v + w^\top \mathbb{E}^\mathbb{Q} [X] \end{aligned} \quad (\text{B.10})$$

$$\begin{aligned} &= \inf_{Z \in R_0(X)} w^\top Z + \sup_{Y \in A_0} w^\top \mathbb{E}^\mathbb{Q} [-Y] + w^\top \mathbb{E}^\mathbb{Q} [X] \\ &= w^\top \mathbb{E}^\mathbb{Q} [-X] + \inf_{Y \in A_0} w^\top \mathbb{E}^\mathbb{Q} [Y] + \sup_{Y \in A_0} w^\top \mathbb{E}^\mathbb{Q} [-Y] + w^\top \mathbb{E}^\mathbb{Q} [X] = 0. \end{aligned} \quad (\text{B.11})$$

Inequality (B.10) follows from the supermartingale property of Theorem 3.1 and (B.11) follows from the choice of (\mathbb{Q}, w) . We now show that $U_t \subseteq G_t^M(w_0^t(\mathbb{Q}, w))$:

$$\begin{aligned} U_t &= \text{cl} [R_t(X) + \beta_t(\mathbb{Q}, w_0^t(\mathbb{Q}, w)) + m_t^X] \\ &= \text{cl} [(R_t(X) + G_t^M(w_0^t(\mathbb{Q}, w))) + (m_t^X + \beta_t(\mathbb{Q}, w_0^t(\mathbb{Q}, w)))] \\ &= \text{cl} [(R_t(X) + G_t^M(w_0^t(\mathbb{Q}, w))) - (G_t^M(w_0^t(\mathbb{Q}, w)) - [m_t^X + \beta_t(\mathbb{Q}, w_0^t(\mathbb{Q}, w))])] \\ &= \text{cl} [(R_t(X) + G_t^M(w_0^t(\mathbb{Q}, w))) - ([-m_t^X + G_t^M(w_0^t(\mathbb{Q}, w))] - \beta_t(\mathbb{Q}, w_0^t(\mathbb{Q}, w)))] \\ &= \text{cl} [(R_t(X) + G_t^M(w_0^t(\mathbb{Q}, w))) - \left(\left[\mathbb{E}^\mathbb{Q} [-X | \mathcal{F}_t] + G_t(w_0^t(\mathbb{Q}, w)) \right] \cap M_t - \beta_t(\mathbb{Q}, w_0^t(\mathbb{Q}, w)) \right)] \\ &\subseteq G_t^M(w_0^t(\mathbb{Q}, w)). \end{aligned} \quad (\text{B.13})$$

Equation (B.12) follows from Proposition B.7 and inclusion (B.13) follows from Proposition B.8. By definition of U_t we obtain $V_t^{(\mathbb{Q}, w_0^t(\mathbb{Q}, w))}(X) = -m_t^X + G_t^M(w_0^t(\mathbb{Q}, w)) = (\mathbb{E}^\mathbb{Q} [-X | \mathcal{F}_t] + G_t(w_0^t(\mathbb{Q}, w))) \cap M_t$ immediately implying by [22, Corollary A.4] that $\left(V_t^{(\mathbb{Q}, w_0^t(\mathbb{Q}, w))}(X) \right)_{t=0}^T$ is a \mathbb{Q} -martingale. It remains to show that $(\mathbb{Q}, w_0^t(\mathbb{Q}, w))$ is a “worst-case” dual pair for any time t . This follows from

$$\begin{aligned} \text{cl} [R_t(X) + G_t^M(w_0^t(\mathbb{Q}, w))] &= V_t^{(\mathbb{Q}, w_0^t(\mathbb{Q}, w))}(X) - \beta_t(\mathbb{Q}, w_0^t(\mathbb{Q}, w)) \\ &= \left(\mathbb{E}^\mathbb{Q} [-X | \mathcal{F}_t] + G_t(w_0^t(\mathbb{Q}, w)) \right) \cap M_t - \beta_t(\mathbb{Q}, w_0^t(\mathbb{Q}, w)). \end{aligned}$$

The first equality follows from Proposition 2.4 (e1) and (e2) from [39] by noting $\beta_t(\mathbb{Q}, w_0^t(\mathbb{Q}, w)) \neq M_t$ and $\beta_t(\mathbb{Q}, w_0^t(\mathbb{Q}, w)) = \emptyset$ would imply $\beta_0(\mathbb{Q}, w) = \emptyset$ by multiperiod time consistency (see Theorem 2.6), which would violate our assumption.

For the converse let $M = \mathbb{R}^d$, if $\left(V_t^{(\mathbb{Q}, w_0^t(\mathbb{Q}, w))}(X) \right)_{t=0}^T$ is a \mathbb{Q} -martingale for some $(\mathbb{Q}, w) \in \mathcal{W}_0$ with $\beta_0(\mathbb{Q}, w) \neq \emptyset$ then the process defined by $U_t := V_t^{(\mathbb{Q}, w_0^t(\mathbb{Q}, w))}(X) +$

$\mathbb{E}^\mathbb{Q}[X|\mathcal{F}_t]$ is one as well. In particular, at the terminal time T , $R_T(X) = R_T(0) - X$ and $U_T = \text{cl}[R_T(0) + \beta_T(\mathbb{Q}, w_0^T(\mathbb{Q}, w))] = G_T(w_0^T(\mathbb{Q}, w))$ by $A_T = R_T(0)$ is a closed and conditionally convex cone (by closed, conditionally convex, and normalized). Since $(U_t)_{t=0}^T$ is a martingale this immediately implies $U_t = G_t(w_0^t(\mathbb{Q}, w))$ by [22, Corollary A.4]. Therefore $\text{cl}[R_t(X) + G_t(w_0^t(\mathbb{Q}, w))] = (\mathbb{E}^\mathbb{Q}[-X|\mathcal{F}_t] + G_t(w_0^t(\mathbb{Q}, w))) - \beta_t(\mathbb{Q}, w_0^t(\mathbb{Q}, w))$ for any time t (utilizing Proposition 2.4(e1) and (e2) from [39]). \square

Proposition B.7. *Let $A, B \in \mathcal{G}(M_t; M_{t,+})$ and $w \in M_{t,+}^+ \setminus M_t^\perp$. Then,*

$$\text{cl}(A + B + G_t^M(w)) = \text{cl}(A + G_t^M(w)) - \cdot (G_t^M(w) - \cdot B).$$

Proof.

$$\begin{aligned} \text{cl}(A + G_t^M(w)) - \cdot (G_t^M(w) - \cdot B) &= \{m \in M_t \mid m + (G_t^M(w) - \cdot B) \subseteq \text{cl}(A + G_t^M(w))\} \\ &= \{m \in M_t \mid m + \{n \in M_t \mid n + B \subseteq G_t^M(w)\} \subseteq \text{cl}(A + G_t^M(w))\} \\ &= \left\{m \in M_t \mid \mathbb{E}[w^\top m] + \right. \\ &\quad \left. \inf \left\{ \mathbb{E}[w^\top n] \mid n \in M_t, \mathbb{E}[w^\top n] + \inf_{b \in B} \mathbb{E}[w^\top b] \geq 0 \right\} \geq \inf_{a \in A} \mathbb{E}[w^\top a] \right\} \\ &= \left\{m \in M_t \mid \mathbb{E}[w^\top m] - \inf_{b \in B} \mathbb{E}[w^\top b] \geq \inf_{a \in A} \mathbb{E}[w^\top a] \right\} \\ &= \left\{m \in M_t \mid \mathbb{E}[w^\top m] \geq \inf_{c \in A+B} \mathbb{E}[w^\top c] \right\} = \text{cl}(A + B + G_t^M(w)). \end{aligned}$$

\square

Proposition B.8. *Let $A, B \in \mathcal{G}(M_t; M_{t,+})$ and $w \in M_{t,+}^+ \setminus M_t^\perp$. If $\text{cl}(A + G_t^M(w)) \subseteq \text{cl}(B + G_t^M(w))$ then $\text{cl}(A + G_t^M(w)) - \cdot \text{cl}(B + G_t^M(w)) \subseteq G_t^M(w)$.*

Proof.

$$\begin{aligned} \text{cl}(A + G_t^M(w)) - \cdot \text{cl}(B + G_t^M(w)) &= \{m \in M_t \mid m + B \subseteq \text{cl}(A + G_t^M(w))\} \\ &= \left\{m \in M_t \mid \mathbb{E}[w^\top m] + \inf_{b \in B} \mathbb{E}[w^\top b] \geq \inf_{a \in A} \mathbb{E}[w^\top a] \right\} \\ &= \left\{m \in M_t \mid \mathbb{E}[w^\top m] \geq \inf_{a \in A} \mathbb{E}[w^\top a] - \inf_{b \in B} \mathbb{E}[w^\top b] \right\} \\ &\subseteq \{m \in M_t \mid \mathbb{E}[w^\top m] \geq 0\} = G_t^M(w). \end{aligned}$$

\square

C Proofs for Section 4

C.1 Conditional scalarization

Proposition C.1. *Let R_t be a c.u.c. and conditionally convex risk measure. Let $w \in \text{recc}(R_t(0))^+ \setminus M_t^\perp$. Then, for every $X \in L^p$*

$$\hat{\rho}_t(X) := \text{ess inf}_{u \in R_t(X)} w^\top u = \text{ess sup}_{(\mathbb{Q}, m_\perp) \in \mathcal{W}_t(w)} \text{ess inf}_{Y \in A_t} (w + m_\perp)^\top \mathbb{E}^\mathbb{Q}[Y - X|\mathcal{F}_t],$$

where $\mathcal{W}_t(w) := \{(\mathbb{Q}, m_\perp) \in \mathcal{M} \times M_t^\perp \mid (\mathbb{Q}, w + m_\perp) \in \mathcal{W}_t\}$.

Proof. Denote the acceptance set by $\hat{A}_t^w := \{X \in L^p \mid \hat{\rho}_t(X) \leq 0\}$. We will show

$$\hat{\rho}_t(X) \geq \operatorname{ess\,sup}_{(\mathbb{Q}, m_\perp) \in \mathcal{W}_t(w)} \operatorname{ess\,inf}_{Y \in \hat{A}_t^w} (w + m_\perp)^\top \mathbb{E}^\mathbb{Q}[Y - X \mid \mathcal{F}_t]. \quad (\text{C.1})$$

The inequality is trivially satisfied if $R_t(X) = \emptyset$ since then $\hat{\rho}_t(X) = \infty$ almost surely. Thus, assume $R_t(X) \neq \emptyset$. Since $\mathbb{E}[\hat{\rho}_t(X)] = \inf_{u \in R_t(X)} \mathbb{E}[w^\top u]$, this implies $\hat{\rho}_t(X) \in L_t^1(\mathbb{R})$. Thus, there exists some $u_X \in M_t$ such that $w^\top u_X = \hat{\rho}_t(X)$. By translativity, this implies that $X + u_X \in \hat{A}_t^w$, which in turn implies

$$(w + m_\perp)^\top \mathbb{E}^\mathbb{Q}[X + u_X \mid \mathcal{F}_t] \geq \operatorname{ess\,inf}_{Y \in \hat{A}_t^w} (w + m_\perp)^\top \mathbb{E}^\mathbb{Q}[Y \mid \mathcal{F}_t]$$

for every $(\mathbb{Q}, m_\perp) \in \mathcal{W}_t(w)$. Subtracting $(w + m_\perp)^\top \mathbb{E}^\mathbb{Q}[X \mid \mathcal{F}_t]$ on both sides of the inequality and then taking the essential supremum over $(\mathbb{Q}, m_\perp) \in \mathcal{W}_t(w)$ yields (C.1). Now we want to show that equality holds in (C.1). In combination with (C.1), we will do this by showing that

$$\mathbb{E}[\hat{\rho}_t(X)] \leq \mathbb{E} \left[\operatorname{ess\,sup}_{(\mathbb{Q}, m_\perp) \in \mathcal{W}_t(w)} \operatorname{ess\,inf}_{Y \in \hat{A}_t^w} (w + m_\perp)^\top \mathbb{E}^\mathbb{Q}[Y - X \mid \mathcal{F}_t] \right]$$

for every $X \in L^p$. By decomposability and Proposition B.1 in conjunction with (B.1) one obtains

$$\mathbb{E}[\hat{\rho}_t(X)] = \sup_{(\mathbb{Q}, m_\perp) \in \mathcal{W}_t(w)} \inf_{Y \in \hat{A}_t^w} \mathbb{E} \left[(w + m_\perp)^\top \mathbb{E}^\mathbb{Q}[Y - X \mid \mathcal{F}_t] \right].$$

Since $\hat{A}_t^w \subseteq A_t^w$ and by decomposability of the necessary sets, the desired result is immediate.

It remains to show that we can replace \hat{A}_t^w with A_t . First note that $\hat{A}_t^w \supseteq A_t$, implying $\hat{\rho}_t(X) \leq \operatorname{ess\,sup}_{(\mathbb{Q}, m_\perp) \in \mathcal{W}_t(w)} \operatorname{ess\,inf}_{Y \in A_t} (w + m_\perp)^\top \mathbb{E}^\mathbb{Q}[Y - X \mid \mathcal{F}_t]$. Let $(u_n)_{n \in \mathbb{N}} \subseteq R_t(X)$ so that $w^\top u_n \searrow \hat{\rho}_t(X)$ w.r.t. almost sure convergence. The existence of such a sequence follows from [28, Theorem A.33(b)] because decomposability of $R_t(X)$ implies the assumption of that theorem. Hence,

$$\begin{aligned} & \operatorname{ess\,sup}_{(\mathbb{Q}, m_\perp) \in \mathcal{W}_t(w)} \operatorname{ess\,inf}_{Y \in A_t} (w + m_\perp)^\top \mathbb{E}^\mathbb{Q}[Y - X \mid \mathcal{F}_t] \\ & \leq \operatorname{ess\,sup}_{(\mathbb{Q}, m_\perp) \in \mathcal{W}_t(w)} \overline{\lim}_{n \rightarrow \infty} (w + m_\perp)^\top \mathbb{E}^\mathbb{Q}[(X + u_n) - X \mid \mathcal{F}_t] \\ & = \operatorname{ess\,sup}_{(\mathbb{Q}, m_\perp) \in \mathcal{W}_t(w)} \overline{\lim}_{n \rightarrow \infty} w^\top u_n = \hat{\rho}_t(X), \end{aligned}$$

where the limit, $\overline{\lim}$, is taken in the almost sure sense. \square

Corollary C.2. *Let $R_{t,s}$ be a c.u.c. and conditionally convex risk measure. Let $w \in \operatorname{recc}(R_t(0))^+ \setminus M_t^\perp$. Then, for every $X \in M_s$*

$$\operatorname{ess\,inf}_{u \in R_t(X)} w^\top u = \operatorname{ess\,sup}_{(\mathbb{Q}, m_\perp) \in \mathcal{W}_{t,s}(w)} \operatorname{ess\,inf}_{Y \in A_{t,s}} (w + m_\perp)^\top \mathbb{E}^\mathbb{Q}[Y - X \mid \mathcal{F}_t],$$

where $\mathcal{W}_{t,s}(w) := \{(\mathbb{Q}, m_\perp) \in \mathcal{M} \times M_t^\perp \mid (\mathbb{Q}, w + m_\perp) \in \mathcal{W}_{t,s}\}$.

Proof. This follows similarly to Proposition C.1. \square

Recall the notation $\mathcal{W}_t^s = \{(\mathbb{Q}, w) \in \mathcal{W}_t \mid \beta_s(\mathbb{Q}, w_t^s(\mathbb{Q}, w)) \neq \emptyset\}$. The next proposition shows that this set coincides with $\widehat{\mathcal{W}}_t^s := \{(\mathbb{Q}, w) \in \mathcal{W}_t \mid \alpha_s(\mathbb{Q}, w_t^s(\mathbb{Q}, w)) \neq \emptyset\}$.

Proposition C.3. $\mathcal{W}_t^s = \widehat{\mathcal{W}}_t^s$ for any times $0 \leq t \leq s \leq T$.

Proof. Let $(\mathbb{Q}, w) \in \mathcal{W}_t^s$, or equivalently $\sup_{Y \in A_s} \mathbb{E}[w_t^s(\mathbb{Q}, w)^\top \mathbb{E}^\mathbb{Q}[-Y \mid \mathcal{F}_s]] < \infty$. By [60, Theorem 1] and A_s decomposable,

$$\sup_{Y \in A_s} \mathbb{E}[w_t^s(\mathbb{Q}, w)^\top \mathbb{E}^\mathbb{Q}[-Y \mid \mathcal{F}_s]] = \mathbb{E}\left[\text{ess sup}_{Y \in A_s} w_t^s(\mathbb{Q}, w)^\top \mathbb{E}^\mathbb{Q}[-Y \mid \mathcal{F}_s]\right].$$

Therefore $\text{ess sup}_{Y \in A_s} w_t^s(\mathbb{Q}, w)^\top \mathbb{E}^\mathbb{Q}[-Y \mid \mathcal{F}_s] \in L_s^1(\mathbb{R})$ and, in particular, there exists some $u \in M_s$ so that $w_t^s(\mathbb{Q}, w)^\top u \geq \text{ess sup}_{Y \in A_s} w_t^s(\mathbb{Q}, w)^\top \mathbb{E}^\mathbb{Q}[-Y \mid \mathcal{F}_s]$ almost surely, i.e., $(\mathbb{Q}, w) \in \widehat{\mathcal{W}}_t^s$, see (A.3). Conversely, let $(\mathbb{Q}, w) \in \widehat{\mathcal{W}}_t^s$ and let $u \in M_s$ so that $u \in \alpha_s(\mathbb{Q}, w_t^s(\mathbb{Q}, w))$. This yields $w_t^s(\mathbb{Q}, w)^\top u \geq \text{ess sup}_{Y \in A_s} w_t^s(\mathbb{Q}, w)^\top \mathbb{E}^\mathbb{Q}[-Y \mid \mathcal{F}_s]$ almost surely. Thus, by [60, Theorem 1] and A_s decomposable,

$$\mathbb{E}[w_t^s(\mathbb{Q}, w)^\top u] \geq \sup_{Y \in A_s} \mathbb{E}[w_t^s(\mathbb{Q}, w)^\top \mathbb{E}^\mathbb{Q}[-Y \mid \mathcal{F}_s]].$$

Therefore $u \in \beta_s(\mathbb{Q}, w_t^s(\mathbb{Q}, w))$ and $(\mathbb{Q}, w) \in \mathcal{W}_t^s$. \square

Remark C.4. By the same logic as the proof of Corollary B.5 and by Proposition C.3, $\text{recc}(R_t(X)) = \bigcap_{(\mathbb{Q}, w) \in \mathcal{W}_t^t} \Gamma_t^M(w)$ for any $X \in L^p$ with $R_t(X) \neq \emptyset$.

C.2 Proof of Corollary 4.1

Proof. We will prove this by showing that the conditional supermartingale property is equivalent to the inclusions

$$R_t(X) \supseteq \bigcup_{Z \in R_s(X)} R_t(-Z) \tag{C.2}$$

$$R_t(X) \subseteq \bigcap_{(\mathbb{Q}, w) \in \mathcal{W}_t} \text{cl} \bigcup_{Z \in R_s(X)} \left[\left(\mathbb{E}^\mathbb{Q}[Z \mid \mathcal{F}_t] + \Gamma_t(w) \right) \cap M_t - \cdot \alpha_{t,s}(\mathbb{Q}, w) \right]. \tag{C.3}$$

Then we will show that (C.2) and (C.3) are equivalent to multiperiod time consistency.

The first part of this proof will be accomplished similarly to Lemma 3.2. We will focus on certain points that are nontrivial to prove. We start with the proof that (C.2) and (C.3) imply the supermartingale property. First, we can see that for $(\mathbb{Q}, w) \in \mathcal{W}_t^t$ (see Proposition C.3), Proposition A.1 yields

$$\begin{aligned} \mathbb{V}_t^{(\mathbb{Q}, w)}(X) &= \left\{ u \in M_t \mid w^\top u \geq \text{ess inf}_{Z \in R_t(X)} w^\top Z + \text{ess inf}_{a_t \in \alpha_t(\mathbb{Q}, w)} -w^\top a_t \right\} \\ &= \left\{ u \in M_t \mid w^\top u \geq \text{ess inf}_{Z \in R_t(X)} w^\top Z + \text{ess sup}_{Y \in A_t} w^\top \mathbb{E}^\mathbb{Q}[-Y \mid \mathcal{F}_t] \right\}. \end{aligned}$$

Now we will show that

$$\begin{aligned} \text{cl } \mathbb{E}^{\mathbb{Q}} \left[\mathbb{V}_s^{(\mathbb{Q}, w_t^s(\mathbb{Q}, w))}(X) \middle| \mathcal{F}_t \right] = \\ \left\{ u \in M_t \mid w^\top u \geq \underset{Z \in R_s(X)}{\text{ess inf}} w^\top \mathbb{E}^{\mathbb{Q}}[Z \mid \mathcal{F}_t] + \underset{Y \in A_s}{\text{ess sup}} w^\top \mathbb{E}^{\mathbb{Q}}[-Y \mid \mathcal{F}_t] \right\}. \end{aligned}$$

“ \subseteq ” The right hand side is closed by the same logic as in Proposition A.1. Furthermore,

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}} \left[\mathbb{V}_s^{(\mathbb{Q}, w_t^s(\mathbb{Q}, w))}(X) \middle| \mathcal{F}_t \right] &= \left\{ \mathbb{E}^{\mathbb{Q}}[u_s \mid \mathcal{F}_t] \mid u_s \in M_s, \right. \\ &\quad \left. w_t^s(\mathbb{Q}, w)^\top u_s \geq \underset{Z \in R_s(X)}{\text{ess inf}} w_t^s(\mathbb{Q}, w)^\top Z + \underset{Y \in A_s}{\text{ess sup}} w_t^s(\mathbb{Q}, w)^\top \mathbb{E}^{\mathbb{Q}}[-Y \mid \mathcal{F}_s] \right\} \\ &\subseteq \left\{ \mathbb{E}^{\mathbb{Q}}[u_s \mid \mathcal{F}_t] \mid u_s \in M_s, \mathbb{E} \left[w_t^s(\mathbb{Q}, w)^\top u_s \middle| \mathcal{F}_t \right] \geq \right. \\ &\quad \left. \mathbb{E} \left[\underset{Z \in R_s(X)}{\text{ess inf}} w_t^s(\mathbb{Q}, w)^\top Z \middle| \mathcal{F}_t \right] + \mathbb{E} \left[\underset{Y \in A_s}{\text{ess sup}} w_t^s(\mathbb{Q}, w)^\top \mathbb{E}^{\mathbb{Q}}[-Y \mid \mathcal{F}_s] \middle| \mathcal{F}_t \right] \right\} \\ &= \left\{ u_t \in M_t \mid w^\top u_t \geq \underset{Z \in R_s(X)}{\text{ess inf}} w^\top \mathbb{E}^{\mathbb{Q}}[Z \mid \mathcal{F}_t] + \underset{Y \in A_s}{\text{ess sup}} w^\top \mathbb{E}^{\mathbb{Q}}[-Y \mid \mathcal{F}_t] \right\}. \end{aligned}$$

Note that we are able to interchange the essential infimum/supremum and conditional expectation due to the decomposability property of $R_s(X)$ and A_s .

“ \supseteq ” By way of contradiction, assume m is an element of the right hand side and $m \notin \text{cl } \mathbb{E}^{\mathbb{Q}} \left[\mathbb{V}_s^{(\mathbb{Q}, w_t^s(\mathbb{Q}, w))}(X) \middle| \mathcal{F}_t \right]$. Since $\text{cl } \mathbb{E}^{\mathbb{Q}} \left[\mathbb{V}_s^{(\mathbb{Q}, w_t^s(\mathbb{Q}, w))}(X) \middle| \mathcal{F}_t \right]$ is closed and convex, we can separate $\{m\}$ from it by some $v \in L_t^q$. That is

$$\begin{aligned} \mathbb{E} \left[v^\top m \right] &< \inf_{u_t \in \text{cl } \mathbb{E}^{\mathbb{Q}} \left[\mathbb{V}_s^{(\mathbb{Q}, w_t^s(\mathbb{Q}, w))}(X) \middle| \mathcal{F}_t \right]} \mathbb{E} \left[v^\top u \right] = \inf_{u_s \in \mathbb{V}_s^{(\mathbb{Q}, w_t^s(\mathbb{Q}, w))}(X)} \mathbb{E} \left[v^\top \mathbb{E}^{\mathbb{Q}}[u_s \mid \mathcal{F}_t] \right] \\ &= \mathbb{E} \left[\underset{u_s \in \mathbb{V}_s^{(\mathbb{Q}, w_t^s(\mathbb{Q}, w))}(X)}{\text{ess inf}} w_t^s(\mathbb{Q}, v)^\top u_s \right], \end{aligned}$$

where in the last equality above we can interchange the expectation and infimum since $\mathbb{V}_s^{(\mathbb{Q}, w_t^s(\mathbb{Q}, w))}(X)$ is decomposable. By construction

$$\underset{u_s \in \mathbb{V}_s^{(\mathbb{Q}, w_t^s(\mathbb{Q}, w))}(X)}{\text{ess inf}} w_t^s(\mathbb{Q}, v)^\top u_s = \begin{cases} \underset{Z \in R_s(X)}{\text{ess inf}} w_t^s(\mathbb{Q}, v)^\top Z & \text{on } D \\ + \underset{Y \in A_s}{\text{ess sup}} w_t^s(\mathbb{Q}, v)^\top \mathbb{E}^{\mathbb{Q}}[-Y \mid \mathcal{F}_s] & \\ -\infty & \text{on } D^C, \end{cases}$$

where $D = \{\omega \in \Omega \mid G_0^M(w_t^s(\mathbb{Q}, v)[\omega]) = G_0^M(w_t^s(\mathbb{Q}, w)[\omega])\}$. Since $\mathbb{Q} \sim \mathbb{P}$, we can conclude that $v(\omega) = \lambda(\omega)w(\omega)$ for some $\lambda \in L_t^0(\mathbb{R}_{++})$ (such that $\lambda w \in L_t^q$). Therefore

$$\mathbb{E} \left[\underset{u_s \in \mathbb{V}_s^{(\mathbb{Q}, w_t^s(\mathbb{Q}, w))}(X)}{\text{ess inf}} w_t^s(\mathbb{Q}, v)^\top u_s \right] > -\infty \text{ if and only if}$$

$$\begin{aligned} \mathbb{E} \left[\underset{u_s \in \mathbb{V}_s^{(\mathbb{Q}, w_t^s(\mathbb{Q}, w))}(X)}{\text{ess inf}} w_t^s(\mathbb{Q}, v)^\top u_s \right] &= \mathbb{E} \left[\lambda \underset{u_s \in \mathbb{V}_s^{(\mathbb{Q}, w_t^s(\mathbb{Q}, w))}(X)}{\text{ess inf}} w_t^s(\mathbb{Q}, w)^\top u_s \right] \\ &= \mathbb{E} \left[\lambda \left(\underset{Z \in R_s(X)}{\text{ess inf}} w_t^s(\mathbb{Q}, w)^\top Z + \underset{Y \in A_s}{\text{ess sup}} w_t^s(\mathbb{Q}, w)^\top \mathbb{E}^{\mathbb{Q}}[-Y \mid \mathcal{F}_s] \right) \right] \\ &= \mathbb{E} \left[\lambda \left(\underset{Z \in R_s(X)}{\text{ess inf}} w^\top \mathbb{E}^{\mathbb{Q}}[Z \mid \mathcal{F}_t] + \underset{Y \in A_s}{\text{ess sup}} w^\top \mathbb{E}^{\mathbb{Q}}[-Y \mid \mathcal{F}_t] \right) \right]. \end{aligned}$$

But this implies

$$\mathbb{E} \left[\lambda w^\top m \right] < \mathbb{E} \left[\lambda \left(\operatorname{ess\,inf}_{Z \in R_s(X)} w^\top \mathbb{E}^\mathbb{Q} [Z | \mathcal{F}_t] + \operatorname{ess\,sup}_{Y \in A_s} w^\top \mathbb{E}^\mathbb{Q} [-Y | \mathcal{F}_t] \right) \right],$$

which is a contradiction to

$$m \in \left\{ u \in M_t \mid w^\top u \geq \operatorname{ess\,inf}_{Z \in R_s(X)} w^\top \mathbb{E}^\mathbb{Q} [Z | \mathcal{F}_t] + \operatorname{ess\,sup}_{Y \in A_s} w^\top \mathbb{E}^\mathbb{Q} [-Y | \mathcal{F}_t] \right\}.$$

The remaining part of the proof that inclusions (C.2) and (C.3) imply the conditional supermartingale property is similar to the proof of Lemma 3.2.

For the reverse implication, let us now assume the conditional supermartingale property. We will prove inclusion (C.2) by showing that the conditional supermartingale property implies time consistency, which then yields (C.2). This proof follows in total analogy to the corresponding proof in Lemma 3.2 since

$$R_t(X) = \bigcap_{(\mathbb{Q}, w) \in \mathcal{W}_t^t} \left[\left(\mathbb{E}^\mathbb{Q} [-X | \mathcal{F}_t] + \Gamma_t(w) \right) \cap M_t - \cdot \alpha_t(\mathbb{Q}, w) \right].$$

Then, one shows that inclusion (C.3) holds, which follows from the same logic as Lemma 3.2 but using the scalarization results from Section C.1 instead. Let us give a short summary of the steps involved.

[21, Lemma 3.18] provides a representation for set-valued dynamic risk measure as an intersection of conditional scalarizations, where one can restrict to $w \in \operatorname{recc}(R_t(0))^+$

$$\begin{aligned} R_{t,s}(Z) &= \bigcap_{w \in \operatorname{recc}(R_t(0))^+ \setminus M_t^\perp} \left\{ m \in M_t \mid w^\top m \geq \hat{\rho}_{t,s}(Z) \right\} \\ &\subseteq \bigcap_{w \in \operatorname{recc}(R_t(0))^+ \setminus M_t^\perp} \left\{ m \in M_t \mid w^\top m \geq \hat{\rho}_t(Z) \right\} = R_t(Z) = R_{t,s}(Z). \end{aligned}$$

Now using the dual representations of the conditional scalarizations $\hat{\rho}_t(Z)$ and $\hat{\rho}_{t,s}(Z)$ from Proposition C.1 and Corollary C.2, we obtain the following since the inclusion above is in fact an equality. For every $Z \in M_s$ and every $w \in \operatorname{recc}(R_t(0))^+ \setminus M_t^\perp$ it holds that for all $(\mathbb{Q}, m_\perp) \in \mathcal{W}_t(w)$

$$\operatorname{ess\,inf}_{Y_{t,s} \in A_{t,s}} (w + m_\perp)^\top \mathbb{E}^\mathbb{Q} [Y_{t,s} - Z | \mathcal{F}_t] \leq \operatorname{ess\,sup}_{(\mathbb{R}, n_\perp) \in \mathcal{W}_t(w)} \operatorname{ess\,inf}_{Y_t \in A_t} (w + n_\perp)^\top \mathbb{E}^\mathbb{R} [Y_t - Z | \mathcal{F}_t].$$

This is because every such constraint is “active” in the above intersection, i.e., if any were made any stricter it would shrink the set $R_{t,s}(Z)$. Because of decomposability of

$$\left\{ \operatorname{ess\,inf}_{Y \in A_t} (w + m_\perp)^\top \mathbb{E}^\mathbb{Q} [Y - X | \mathcal{F}_t] \mid (\mathbb{Q}, m_\perp) \in \mathcal{W}_t(w) \right\},$$

there exists a monotonically increasing sequence that converges to the essential supremum almost surely. Now consider the above inequality for the case $m_\perp = 0 \in M_t^\perp$ and also note that for $w \notin \operatorname{recc}(R_t(0))^+ \setminus M_t^\perp$, $\operatorname{ess\,inf}_{Y_{t,s} \in A_{t,s}} (w + m_\perp)^\top \mathbb{E}^\mathbb{Q} [Y_{t,s} - Z | \mathcal{F}_t] = -\infty$ for any $(\mathbb{Q}, m_\perp) \in \mathcal{W}_t(w)$. Then, we can conclude for every $Z \in M_s$ it holds that

for all $(\mathbb{Q}, w) \in \mathcal{W}_t$ there exists a sequence $(\mathbb{R}_k, v_k)_{k \in \mathbb{N}} \subseteq \mathcal{W}_t$ such that $v_k \in w + M_t^\perp$ for every k and

$$\operatorname{ess\,inf}_{Y_{t,s} \in A_{t,s}} w^\top \mathbb{E}^\mathbb{Q} [Y_{t,s} - Z | \mathcal{F}_t] \leq \overline{\lim}_{k \rightarrow \infty} \operatorname{ess\,inf}_{Y_t \in A_t} v_k^\top \mathbb{E}^{\mathbb{R}_k} [Y_t - Z | \mathcal{F}_t],$$

where $\overline{\lim}$ indicates the almost sure limit. In particular, for every $(\mathbb{Q}, w) \in \mathcal{W}_t$ there exists some sequence $\mathbb{R}_k(\mathbb{Q}, w, Z) \in \mathcal{M}$ and $v_k(\mathbb{Q}, w, Z) \in w + M_t^\perp$ such that $(\mathbb{R}_k(\mathbb{Q}, w, Z), v_k(\mathbb{Q}, w, Z))_{k \in \mathbb{N}} \subseteq \mathcal{W}_t$ and

$$\begin{aligned} & \left(\mathbb{E}^\mathbb{Q} [-Z | \mathcal{F}_t] + \Gamma_t(w) \right) \cap M_t - \cdot \alpha_{t,s}(\mathbb{Q}, w) \supseteq \\ & \overline{\text{cl}} \bigcup_{k \in \mathbb{N}} \left[\left(\mathbb{E}^{\mathbb{R}_k(\mathbb{Q}, w, Z)} [-Z | \mathcal{F}_t] + \Gamma_t(v_k(\mathbb{Q}, w, Z)) \right) \cap M_t - \cdot \alpha_t(\mathbb{R}_k(\mathbb{Q}, w, Z), v_k(\mathbb{Q}, w, Z)) \right], \end{aligned}$$

where $\overline{\text{cl}}$ indicates the almost sure closure. This implies the desired inclusion (C.3):

$$\begin{aligned} R_t(X) & \subseteq \bigcap_{(\mathbb{Q}, w) \in \mathcal{W}_t} \text{cl} \bigcup_{Z \in R_s(X)} \left[\left(\mathbb{E}^\mathbb{Q} [Z | \mathcal{F}_t] + \Gamma_t(w) \right) \cap M_t - \cdot \alpha_t(\mathbb{Q}, w) \right] \\ & \subseteq \bigcap_{(\mathbb{Q}, w) \in \mathcal{W}_t} \text{cl} \bigcup_{Z \in R_s(X)} \overline{\text{cl}} \bigcup_{k \in \mathbb{N}} \left[\left(\mathbb{E}^{\mathbb{R}_k(\mathbb{Q}, w, -Z)} [Z | \mathcal{F}_t] + \Gamma_t(v_k(\mathbb{Q}, w, -Z)) \right) \cap M_t \right. \\ & \quad \left. - \cdot \alpha_t(\mathbb{R}_k(\mathbb{Q}, w, -Z), v_k(\mathbb{Q}, w, -Z)) \right] \\ & \subseteq \bigcap_{(\mathbb{Q}, w) \in \mathcal{W}_t} \text{cl} \bigcup_{Z \in R_s(X)} \left[\left(\mathbb{E}^\mathbb{Q} [Z | \mathcal{F}_t] + \Gamma_t(w) \right) \cap M_t - \cdot \alpha_{t,s}(\mathbb{Q}, w) \right]. \end{aligned}$$

The last part of the proof is to show that (C.2) and (C.3) are equivalent to multiperiod time consistency. It is trivially true that multiperiod time consistency implies the inclusions (C.2) and (C.3) by the recursive formulation in Theorem 2.5, using for (C.3) that the union of intersections is contained in intersection of unions. For the converse implication we can use the same logic as in the proof of Lemma 3.3 to show that (C.2) and (C.3) are equivalent to

$$A_t \supseteq A_s + A_{t,s} \tag{C.4}$$

$$A_t \subseteq \bigcap_{(\mathbb{Q}, w) \in \mathcal{W}_t} [A_s + \text{cl} (A_{t,s} + \Gamma_s^M(w_t^s(\mathbb{Q}, w)))] \tag{C.5}$$

when $(R_t)_{t=0}^T$ is conditionally c.u.c. By $\Gamma_s^M(w_t^s(\mathbb{Q}, w)) \subseteq G_s^M(w_t^s(\mathbb{Q}, w))$, (C.4) and (C.5) imply (3.4) and (3.5), which imply multiperiod time consistency by Theorem 3.1 and Lemmas 3.2 and 3.3. \square

C.3 Proof of Corollary 4.3

Proof. Most aspects of the proof are similar to the proof of Corollary 3.5, using a straight forward extension of Proposition 2.4 (e1) and (e2) from [39] for conditionally convex and decomposable sets and the trivial conditional versions of Proposition B.7 and B.8 (given in Proposition C.5). We will here only show $\mathbb{U}_t = \Gamma_t^M(w_0^t(\mathbb{Q}, w))$ as the proof differs slightly from the proof of Corollary 3.5. Define $m_t^X \in M_t$ such that

$$m_t^X + \Gamma_t^M(w_0^t(\mathbb{Q}, w)) = \left(\mathbb{E}^\mathbb{Q} [X | \mathcal{F}_t] + \Gamma_t(w_0^t(\mathbb{Q}, w)) \right) \cap M_t$$

and define $\mathbb{U}_t := \mathbb{V}_t^{(\mathbb{Q}, w_0^t(\mathbb{Q}, w))}(X) + m_t^X$ for any time t . Since $\Gamma_0^M = G_0^M$ by definition, one obtains $\mathbb{U}_0 = \Gamma_0^M(w)$ from the proof of Corollary 3.5. Similar to the proof of Corollary 3.5, we can show that $\mathbb{U}_t \subseteq \Gamma_t^M(w_0^t(\mathbb{Q}, w))$ for any time t . For the reverse, we use the fact that \mathbb{U}_t defines a supermartingale, which follows from the properties of $\mathbb{V}_t^{(\mathbb{Q}, w_0^t(\mathbb{Q}, w))}(X)$ and m_t^X . Thus, $\mathbb{U}_t \subseteq \text{cl } \mathbb{E}[\mathbb{U}_s | \mathcal{F}_t]$. Therefore, for any time t we obtain

$$\Gamma_0^M(w) = \mathbb{U}_0 \subseteq \text{cl } \mathbb{E}^\mathbb{Q}[\mathbb{U}_t] \subseteq \text{cl } \mathbb{E}^\mathbb{Q}[\Gamma_t^M(w_0^t(\mathbb{Q}, w))] = \Gamma_0^M(w)$$

by [22, Corollary A.6]. Let us assume $\mathbb{U}_t \subsetneq \Gamma_t^M(w_0^t(\mathbb{Q}, w))$, i.e. there exists some $\delta > 0$ such that $\mathbb{P}(\text{ess inf}_{u \in \mathbb{U}_t} w_0^t(\mathbb{Q}, w)^\top u \geq \delta) > 0$. However, this contradicts $\text{cl } \mathbb{E}^\mathbb{Q}[\mathbb{U}_t] = \Gamma_0^M(w)$ since

$$0 = \inf_{u_0 \in \text{cl } \mathbb{E}^\mathbb{Q}[\mathbb{U}_t]} w^\top u = \inf_{u_t \in \mathbb{U}_t} \mathbb{E} \left[w_0^t(\mathbb{Q}, w)^\top u_t \right] \geq \delta \mathbb{P}(\text{ess inf}_{u_t \in \mathbb{U}_t} w_0^t(\mathbb{Q}, w)^\top u_t) > 0.$$

□

Proposition C.5. *Let $A, B \in \mathcal{G}(M_t; M_{t,+})$ conditionally convex and $w \in M_{t,+}^+ \setminus M_t^\perp$. Then*

$$\text{cl}(A + B + \Gamma_t^M(w)) = \text{cl}(A + \Gamma_t^M(w)) - \cdot (\Gamma_t^M(w) - \cdot B).$$

Additionally, if $\text{cl}(A + \Gamma_t^M(w)) \subseteq \text{cl}(B + \Gamma_t^M(w))$ then

$$\text{cl}(A + \Gamma_t^M(w)) - \cdot \text{cl}(B + \Gamma_t^M(w)) \subseteq \Gamma_t^M(w).$$

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